

DYNAMIC BEHAVIOUR OF UNSATURATED POROUS MEDIA: GOVERNING EQUATIONS USING THE THEORY OF MIXTURES WITH INTERFACES (TMI)

KANTHASAMY K. MURALEETHARAN* AND CHANGFU WEI

School of Civil Engineering and Environmental Science, University of Oklahoma, Norman, OK 73019, U.S.A.

SUMMARY

Theory of Mixtures with Interfaces (TMI) is used to develop field equations governing the behaviour of unsaturated porous media under dynamic loading conditions. Interfaces existing between bulk phases in unsaturated porous media are explicitly considered in the TMI. Volume fractions and area densities are introduced as independent constitutive variables. A new definition for the total stress that explicitly includes interfacial effects is deduced. Thermodynamic restrictions are used to derive the constitutive relationships. It is found that there exists a well-defined potential (i.e. Gibbs' thermodynamic potential) to drive the flow of the fluids. Effective stress for unsaturated soils is theoretically shown to be dependent not only on the degree of saturation but also on the stress path and soil type. Theoretical basis for the use of two stress state variables, net stress and suction, instead of a single effective stress is provided. The soil–water characteristic curve is shown to be none other than the linear momentum balance equation of all the interfaces. Realistic functional forms are suggested to take into account the variation of suction when the solid skeleton undergoes deformation. The final set of governing equations presented are easily implementable into finite element methods and can be used to analyse problems such as the earthquake loading of compacted soil embankments. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: unsaturated soil; theory of mixtures; interfaces; dynamics; stress state; thermodynamic potential

INTRODUCTION

Porous media like soils have been viewed as mixtures consisting of a solid skeleton and certain numbers of immiscible fluids, e.g. water, oil, or gas. According to the degree of saturation of the specified fluid, the porous media may be categorized into three major groups: saturated, dry, and unsaturated. The static and dynamic behaviour of the saturated soils have been extensively investigated,^{1–6} and the success of these studies is so remarkable that it could be claimed that the behaviour of the saturated porous media has been well understood. In principle, if the fluid in the soil is considered as compressible, the dry porous media may be treated in the ways similar to those applied in the saturated case. In contrast, the study of the behaviour of the unsaturated porous media has been lagging far behind its saturated counterpart.

*Correspondence to: Kanthasamy K. Muraleetharan, School of Civil Engineering and Environmental Science, University of Oklahoma, 202 W. Boyd St, RM 334, Norman, OK 73019-1024, U.S.A. E-mail: muralee@ou.edu

Contract/grant sponsor: National Science Foundation; Contract/grant number: CMS 9501718

The unsaturated porous media are those consisting of a solid and a certain number of fluid phases. A salient feature of the unsaturated porous media is the existence of interfacial effects between the fluid phases. It has been argued that the interface between the liquid and the gas, called contractile skin and characterized by surface tension, may act as an independent phase^{7,8} and may therefore affect the deformation behaviour and the strength of the unsaturated soils.⁹ Therefore, a fully coupled analysis of unsaturated porous media requires a procedure that can rigorously incorporate the interfaces and all the interactions among these interfaces and bulk phases. Thus far, realistic field equations governing the dynamic behaviour of unsaturated porous media have not been developed. This lacuna stems from the great difficulties involved in the evaluation of the interfacial effects.

The development of the theory of mixtures^{10,11} provides a rational way for constructing more general and realistic models to describe the static and dynamic behaviour of unsaturated porous media. Traditionally, in the derivation of this approach, balance equations for independent phases are introduced, and the constitutive relations for the unknown properties are developed by the principles of the mixture theory, among which is the second law of thermodynamics. By incorporating these constitutive relations, governing equations are established and the unknown variables can be solved. Thus far, almost all the works on this subject are only concerned with the mixtures consisting of a linear elastic or rigid solid skeleton and certain numbers of fluids.^{11–16} No considerations have been given to interfaces or inelastic behaviour of the solid skeleton in the above-mentioned works. Actually, even for the mixture with a linear elastic solid skeleton, a large number of unknown constitutive parameters, which are extremely difficult to evaluate, are included in the proposed governing equations. After scrutinizing the available publications, we found that very few published works have given effective physical explanations of the constitutive parameters included in their governing equations. On the other hand, earlier cited works^{3,5,6} on saturated porous media were successful in including elastoplastic behaviour of the solid skeleton through phenomenological constitutive models.

Due to the success of effective stress concept in the saturated soil mechanics, many efforts have been made to incorporate the effective stress concept into the unsaturated soil problem.^{17–21} In this approach, the balance equations at macroscopic level are introduced, and by the incorporation of the effective stress concept a phenomenological constitutive relation for the solid skeleton is established. This procedure seem to be promising for the analysis of nearly saturated porous media, where the interfacial effect can be neglected and the effect of the degree of saturation on the behaviour of the porous media can be implicitly and approximately taken into account by defining the fluid compressibility as a function of the degree of saturation.

Based on an averaging procedure, Shrefler *et al.*²² developed an analysis procedure, which has been extensively used in the static and dynamic analysis of the behaviour of unsaturated soils with various degrees of saturation.^{23–28} The effective stress formula used by Shrefler *et al.*²² is basically the same as that proposed by Bishop²⁹ with $\chi = S_r$, where S_r is the degree of saturation and χ is the only unknown parameter in Bishop's formulation. Bishop considered χ to be a function of S_r . Sufficient experimental results show that χ cannot be uniquely determined by S_r , instead, it is a function of S_r , the soil type and stress history. These results imply that the calculation of effective stress for unsaturated soils involves great uncertainties. Furthermore, a so-called 'soil-water characteristic curve',³⁰ which was borrowed from soil science, has been incorporated in all the aforementioned analyses.^{23–28} Thus far, the mechanics rationale behind the use of this relationship is not clear.

Arguing that stress state variable should be independent of soil properties, Fredlund and Morgenstern⁸ suggested that, instead of a single effective stress state variable, two stress state variables, i.e. net stress (total stress minus pore gas pressure) and matric suction (pore gas pressure minus pore liquid pressure), should be introduced to describe the behaviour of unsaturated soils. They considered the interface between the liquid and the gas called 'contractile skin' as an independent phase in their analysis. However, Fredlund and Morgenstern's⁸ theoretical justification for including the two stress state variables failed to properly address the scale issue. Suction values calculated using microscopic capillary phenomenon were directly introduced into macroscopic equilibrium equations. Although the theoretical basis for using the two stress state variables is weak, it has been used to solve practical problems involving unsaturated soils³⁰ such as shear strength, seepage, and volume change in an uncoupled manner. In recent years, the use of two stress state variables has proven to be very effective in describing the constitutive behaviour of unsaturated soil through phenomenological means.^{31,32}

In a parallel front, the study of multiphase flows is encouraging. The balance equations for the interfaces as well as bulk phases have been obtained by the local averaging procedure.^{33,34} These developments make it possible for us to construct realistic governing equations that describes the behaviour of unsaturated porous media based on the theory of mixtures with the inclusion of interfacial effects. In order to reflect this explicit consideration of interfaces, we will refer to the above mentioned works as the *Theory of Mixtures with Interfaces (TMI)*. In this paper, our primary motivation is a desire to develop governing equations for dynamic behaviour of unsaturated porous media based on the two stress state variables. To fulfil this goal, the thermodynamic basis for the use of the two stress state variables will be pursued and the possible structure of the constitutive relations for the unsaturated porous media will be investigated based on the TMI. To include the interfacial effect explicitly, the interface between the liquid and the gas (i.e. the contractile skin) will be assigned the thermodynamic properties, e.g. mass and surface tension. Contractile skin will be considered as an independent phase participating in the superposition of various phases to form the overall behaviour of the mixture. For the sake of simplicity, and without losing the generality, in a soil mechanics oriented theory, the unsaturated porous media are considered to be composed of a solid phase, a liquid and a gas. Present work differs from those in previous publications in three important aspects. First, volume fractions of the bulk phases and area densities of interfaces are introduced as independent variables into constitutive relations. We consider volume fractions and area densities to be more fundamental parameters than degree of saturation and porosity used by some of the previous works.^{35,36} Second, a thermodynamic basis for the use of two stress state variables is presented. In doing so an expression for total stress in unsaturated soils is derived which clearly shows the influence of interfacial effects on the overall behaviour. Finally, it is shown that the so called 'soil-water characteristic curve' is none other than the linear momentum balance equation of all the interfaces present in an unsaturated soil. The final set of governing equations presented are easily implementable into numerical procedures such as finite element method and together with phenomenological elastoplastic constitutive models³² can be used to analyse behaviour of structures such as compacted soil embankments subjected to earthquake loadings.

BASIC CONCEPTS AND KINEMATICS

Throughout this paper, the unsaturated porous medium is viewed as a mixture consisting of a deformable solid skeleton, a liquid and a gas, designated by s , ℓ , and g , respectively. Unless

specified, the Greek letters α , β , or γ represent all the three bulk phases, i.e. s, ℓ , and g, and their combinations, e.g. $\alpha\beta$, designate all the three interfaces, $s\ell$, ℓg , and sg ; it is understood that $\alpha\beta$ and $\beta\alpha$ represent the same interface. Greek letter λ designates all the phases including interfaces. Obviously, $s\ell g$ (i.e. $\alpha\beta\gamma$) means the contact line between all the three interfaces, which is assumed to possess no thermodynamic properties. Standard solid mechanics sign convention is used, i.e. tensile stress is positive, in presenting the equations. Bold characters are used to represent tensors.

In unsaturated porous media, at least there exist three distinct scales of motions: molecular, microscopic (pore scale), and macroscopic. On the pore scale, the continuum theory of physics is applicable within each phase and interface. In practice, of main concern are the collective properties of the phases and the mixture at macroscopic scale. For this purpose, the classical continuum theory of mixtures by Truesdell and Toupin¹⁰ metaphysically assumes that the constituents of a mixture could be modelled as superimposed continua such that each point in the mixture is simultaneously occupied by a material point of each constituent (phase). On the other hand, it has been argued that³⁵ some important effects, e.g. the effect of interfaces, may be overlooked by the classical theory of mixtures. An alternative approach used to obtain the averaged properties of the phases and the mixture is the local volume averaging procedure by Hassanizadeh and Gray³³ and Gray and Hassanizadeh.³⁴ This procedure will be followed here.

It is assumed that there exists an averaging volume or area with a characteristic length D ($\ell \ll D \ll L$), over which the properties of the bulk phases and the interfaces averaged will keep practically constant during the small variation of the size of the averaging volume. Here, ℓ is the microscopic scale of the medium and L is the scale of the gross inhomogeneity. Typically, $\ell = 50 \mu\text{m}$ in sands and $\ell = 1 \mu\text{m}$ in clay, whereas $L = 1 \text{ cm}$.³⁷ The averaging volume, denoted by δV , is called a Representative Elementary Volume (REV). The averaging volume is defined such that its shape, size, and orientation are identical at all the locations and for all the time. All the averaged properties and balance equations of bulk and interfacial phases are assigned to the centre of the REV. Since each point in the domain of interest can be considered as the centre of some averaging volume, the averaged properties may be considered spatially continuous. By averaging microscopic balance equations over a REV, macroscopic equations valid at every point of the domain are established.

The concept of volume fraction is formulated for the porous solid, the liquid and the gas. The actual volume δV of a REV with a surface δA is composed additively of the partial volume δV^α of α -bulk phase,

$$\delta V = \sum_{\alpha} \delta V^{\alpha} \quad (1)$$

The volume fraction of α -phase, n^α , is defined by

$$n^\alpha(\mathbf{x}, t) = \delta V^\alpha / \delta V \quad (2)$$

n^α is a locally averaged property and has been assigned to the centre of the REV, and therefore is a continuous function of the spatial location \mathbf{x} and time t . The same argument applies to other quantities, such as specific area $a^{\alpha\beta}$, density ρ^α , etc., which will be presented below. From (1) and (2), it can be shown n^α is constrained by

$$\sum_{\alpha} n^\alpha(\mathbf{x}, t) = 1 \quad (3a)$$

$$0 \leq n^\alpha(\mathbf{x}, t) \leq 1 \quad (3b)$$

The specific area (or area density) of $\alpha\beta$ -interface, $a^{\alpha\beta}$, is the area of $\alpha\beta$ -interface per unit total volume of the REV,

$$a^{\alpha\beta}(\mathbf{x}, t) = \delta A_{\alpha\beta} / \delta V \quad (4)$$

where $\delta A_{\alpha\beta}$ is the total area of $\alpha\beta$ -interface within the REV.

From definitions (1)–(4), it is readily understood that quantities, n^α and $a^{\alpha\beta}$, will directly reflect the deformation behaviour of the bulk phases, the interfaces and the mixture and the interactions between the bulk phases. Therefore, these two properties will serve as important independent variables in our theory.

The mass of the α -phase, i.e. δm^α , in the REV is given by

$$\delta m^\alpha = \rho^\alpha \delta V^\alpha = \rho^\alpha n^\alpha \delta V = \rho_\alpha \delta V \quad (5)$$

where ρ_α and ρ^α are the macroscopic (volume averaged) partial density and the intrinsic density of α -phase, respectively, and $\rho_\alpha = n^\alpha \rho^\alpha$. If α -phase is incompressible, ρ^α will be a constant. Although all the interfaces are considered mathematically singular (i.e. zero volume) as implied by equation (3a), they do have finite thicknesses from a physical point of view. To reconcile this inconsistency, the concept of excess interfacial quantities is introduced, which allows us to define the surface (excess) density of various properties for the interfaces (the interested reader may consult Miller and Neogi³⁸ for a detailed account). Therefore, we have the following definition of the volume averaged surface mass density:

$$\Gamma^{\alpha\beta} = \frac{1}{\delta A_{\alpha\beta}} \int_{\delta A_{\alpha\beta}} \Gamma \Big|_{\alpha\beta} da = \frac{1}{a^{\alpha\beta} \delta V} \int_{\delta A_{\alpha\beta}} \Gamma \Big|_{\alpha\beta} da \quad (6)$$

where $\Gamma|_{\alpha\beta}$ is the microscopic excessive mass of $\alpha\beta$ -interface.³⁵ The overall density of the mixture is obtained by

$$\rho(\mathbf{x}, t) = \sum_{\alpha} n^\alpha \rho^\alpha + \sum_{\alpha\beta} a^{\alpha\beta} \Gamma^{\alpha\beta} \quad (7)$$

The superposition of bulk and interfacial phases implies that a spatial point \mathbf{x} in the domain of interest could be simultaneously occupied by the material particles from each independent phase, which is characterized by an initial material position \mathbf{X}^λ . Thus, the motion of all the material points is expressed by

$$\mathbf{x} = \mathbf{x}^\lambda(\mathbf{X}^\lambda, t) \quad (8)$$

Deformation gradient of the skeleton is defined by

$$\mathbf{F}^s = \partial \mathbf{x}^s / \partial \mathbf{X} = \text{GRAD } \mathbf{x}^s \quad (9)$$

Because all the averaged properties and the associated relationships (e.g. macroscopic balance equations) have been assigned to the centre of the averaging volume (REV), \mathbf{x} , the motions of various bulk phases and interfaces are understood as the time-dependent sequences of placements. Assuming the usual continuity and differentiability of \mathbf{x}^λ , we define the velocity and acceleration vectors of the λ -phase

$$\mathbf{v}^\lambda(\mathbf{x}, t) = \dot{\mathbf{x}}^\lambda = D^\lambda \mathbf{x} / Dt \quad (10a)$$

$$\mathbf{a}^\lambda(\mathbf{x}, t) = \ddot{\mathbf{x}}^\lambda = D^{\lambda 2} \mathbf{x}^\lambda / Dt^2 \quad (10b)$$

D^λ/Dt is the material derivative with respect to the motion of λ -phase, and

$$\frac{D^\lambda}{Dt}(\cdot) = \frac{\partial}{\partial t}(\cdot) + \mathbf{v}^\lambda \cdot \nabla(\cdot) = \frac{D^\alpha}{Dt}(\cdot) + \mathbf{v}^{\lambda,\alpha} \cdot \nabla(\cdot) \quad (11)$$

where

$$\mathbf{v}^{\lambda,\alpha} = \mathbf{v}^\lambda - \mathbf{v}^\alpha \quad (12)$$

is the relative velocity of λ -phase with respect to α -phase. The symbol $\nabla (= \partial/\partial \mathbf{x})$ always represents gradients with respect to \mathbf{x} in this paper. In the following derivations, Lagrangian strain tensor \mathbf{E}^s of the solid skeleton, a macroscopic quantity, will be introduced, that is

$$\mathbf{E}^s = \frac{1}{2}(\mathbf{F}^{sT} \cdot \mathbf{F}^s - \mathbf{I}) \quad (13)$$

Finally, the symmetric parts of the velocity gradients, \mathbf{d}^α and $\mathbf{d}^{\alpha\beta}$, are defined by

$$\mathbf{d}^\alpha = \frac{1}{2}[\nabla \mathbf{v}^\alpha + (\nabla \mathbf{v}^\alpha)^T] \quad (14)$$

$$\mathbf{d}^{\alpha\beta} = \frac{1}{2}[\nabla \mathbf{w}^{\alpha\beta} + (\nabla \mathbf{w}^{\alpha\beta})^T] \quad (15)$$

where \mathbf{v}^α and $\mathbf{w}^{\alpha\beta}$ are the velocities of α -phase and $\alpha\beta$ -interface, respectively.

BALANCE EQUATIONS

Macroscopic balance equations of bulk phases and interfaces have been obtained by using the local averaging procedure.^{3,5} In this section, these equations will be introduced directly.

Balance equations of mass

For a bulk phase

$$\frac{D^\alpha(n^\alpha \rho^\alpha)}{Dt} + n^\alpha \rho^\alpha (\nabla \cdot \mathbf{v}^\alpha) = \sum_{\beta \neq \alpha} \hat{e}_{\alpha\beta}^\alpha \quad (16a)$$

For an interface

$$\frac{D^{\alpha\beta}}{Dt}(a^{\alpha\beta} \Gamma^{\alpha\beta}) + a^{\alpha\beta} \Gamma^{\alpha\beta} (\nabla \cdot \mathbf{w}^{\alpha\beta}) = -\hat{e}_{\alpha\beta}^\alpha - \hat{e}_{\alpha\beta}^\beta + \hat{e}_{s/g}^{\alpha\beta} \quad (16b)$$

The quantities $\hat{e}_{\alpha\beta}^\alpha$ in the right-hand side (r.h.s.) of (16a) account for the rate of mass transfer from $\alpha\beta$ -interface to α -bulk phase. Throughout this paper, a carat implies an exchange term. Similarly, the mass exchange term $\hat{e}_{s/g}^{\alpha\beta}$ represents the rate of mass exchange from contact line s/g to $\alpha\beta$ -interface. For convenience, the notations are summarized in the nomenclature section at the end of the paper. Equation (16b) is subject to

$$\sum_{\alpha\beta} \hat{e}_{s/g}^{\alpha\beta} = 0 \quad (16c)$$

This restriction follows from the assumption that the contact line possesses no thermodynamic properties, i.e. no excess mass.

Balance equations of linear momentum

For a bulk phase

$$n^\alpha \rho^\alpha \frac{D^\alpha \mathbf{v}^\alpha}{Dt} - \nabla \cdot (n^\alpha \mathbf{t}^\alpha) - n^\alpha \rho^\alpha \mathbf{g}^\alpha = \sum_{\beta \neq \alpha} \hat{\mathbf{T}}_{\alpha\beta}^\alpha \quad (17a)$$

For an interface

$$\begin{aligned} a^{\alpha\beta} \Gamma_{\alpha\beta} \frac{D^{\alpha\beta} \mathbf{w}^{\alpha\beta}}{Dt} - \nabla \cdot (a^{\alpha\beta} \mathbf{S}^{\alpha\beta}) - a^{\alpha\beta} \Gamma_{\alpha\beta} \mathbf{g}^{\alpha\beta} \\ = -(\hat{\mathbf{T}}_{\alpha\beta}^\alpha + \hat{e}_{\alpha\beta}^\alpha \mathbf{v}^{\alpha,s}) - (\hat{\mathbf{T}}_{\alpha\beta}^\beta + \hat{e}_{\alpha\beta}^\beta \mathbf{v}^{\beta,s}) + (\hat{e}_{\alpha\beta}^\alpha + \hat{e}_{\alpha\beta}^\beta) \mathbf{w}^{\alpha\beta,s} + \hat{\mathbf{S}}_{s/g}^{\alpha\beta} \end{aligned} \quad (17b)$$

In (17) \mathbf{t}^α and $\mathbf{S}^{\alpha\beta}$ account for the general flux vector of momentum in α -phase and on $\alpha\beta$ -interface, respectively, which are also understood as macroscopic stress tensors. The quantity $\mathbf{S}^{\alpha\beta}$ is related to the macroscopic surface tension. The quantities $n^\alpha \mathbf{t}^\alpha$ and $a^{\alpha\beta} \mathbf{S}^{\alpha\beta}$ are called partial stress tensors. Again, for the sake of convenience, notations are summarized in the nomenclature section at the end of the paper, and will not be explained here. Because the contact line has no thermodynamic properties, (17b) is restricted by

$$\sum_{\alpha\beta} (\hat{\mathbf{S}}_{s/g}^{\alpha\beta} + \hat{e}_{s/g}^{\alpha\beta} \mathbf{w}^{\alpha\beta}) = 0 \quad (17c)$$

We assume that the gravity is the only source of body force. Thus, the external supply of momentum \mathbf{g}^α and $\mathbf{g}^{\alpha\beta}$ are equal to each other in the problem of concern, i.e. $\mathbf{g}^\alpha = \mathbf{g}^{\alpha\beta} = \mathbf{g}$, \mathbf{g} is the gravity vector.

Summing (17a) over all the bulk phases and (17b) over all the interfaces and then summing these resulting equations together, we obtain the overall linear momentum balance of the mixture as follows:

$$\begin{aligned} \sum_\alpha n^\alpha \rho^\alpha \frac{D^\alpha \mathbf{v}^\alpha}{Dt} + \sum_{\alpha\beta} a^{\alpha\beta} \Gamma_{\alpha\beta} \frac{D^{\alpha\beta} \mathbf{w}^{\alpha\beta}}{Dt} - \nabla \cdot \mathbf{t} - \rho \mathbf{g} \\ = \sum_\alpha \sum_{\beta \neq \alpha} \hat{e}_{\alpha\beta}^\alpha \mathbf{w}^{\alpha\beta,\alpha} - \sum_{\alpha\beta} \hat{e}_{s/g}^{\alpha\beta} \mathbf{w}^{\alpha\beta} \end{aligned} \quad (18)$$

where ρ is the overall density of the mixture given by (7) and \mathbf{t} is the total stress of the mixture defined as

$$\mathbf{t} = \sum_\alpha n^\alpha \mathbf{t}^\alpha + \sum_{\alpha\beta} a^{\alpha\beta} \mathbf{S}^{\alpha\beta} \quad (19)$$

Equation (19) provides, for the first time, an expression for total stress incorporating the influence of all the bulk phases and interfaces in a mixture. This expression will be later used in justifying the use of two stress state variables in unsaturated soil mechanics. If the interfacial effects are ignored, i.e. if $a^{\alpha\beta} \mathbf{S}^{\alpha\beta}$ terms are neglected, (19) will become

$$\mathbf{t} = \sum_\alpha \mathbf{t}_\alpha = \sum_\alpha n^\alpha \mathbf{t}^\alpha \quad (20)$$

Several previous researchers²³⁻²⁷ have used (20) in analysing the behaviour of unsaturated soils. In other words, interfacial effects were not considered in their analyses.

Balance of moment of momentum

We assume that all the phases in the mixture are microscopically non-polar. The balance law of angular momentum reveals that stress tensor \mathbf{t}^α and $\mathbf{S}^{\alpha\beta}$ are symmetric, i.e.

$$\mathbf{t}^\alpha = (\mathbf{t}^\alpha)^T \quad (21a)$$

and

$$\mathbf{S}^{\alpha\beta} = (\mathbf{S}^{\alpha\beta})^T \quad (21b)$$

By introducing the definition (19), one may prove that the total stress \mathbf{t} is also symmetric, i.e.

$$\mathbf{t} = \mathbf{t}^T \quad (22)$$

This result proves extremely useful in restricting the constitutive forms of the partial stress tensors.

Balance equations of energy

For a bulk phase

$$n^\alpha \rho^\alpha \frac{D^\alpha E^\alpha}{Dt} - n^\alpha \mathbf{t}^\alpha : \mathbf{d}^\alpha - \nabla \cdot (n^\alpha \mathbf{q}^\alpha) - n^\alpha \rho^\alpha h^\alpha = \sum_{\beta \neq \alpha} \hat{Q}_{\alpha\beta}^\alpha \quad (23a)$$

For an interface

$$\begin{aligned} a^{\alpha\beta} \Gamma^{\alpha\beta} \frac{D^{\alpha\beta} E^{\alpha\beta}}{Dt} - a^{\alpha\beta} \mathbf{S}^{\alpha\beta} : \mathbf{d}^{\alpha\beta} - \nabla \cdot (a^{\alpha\beta} \mathbf{q}^{\alpha\beta}) - a^{\alpha\beta} \Gamma^{\alpha\beta} h^{\alpha\beta} \\ = -[\hat{Q}_{\alpha\beta}^\alpha + \hat{\mathbf{T}}_{\alpha\beta}^\alpha \cdot \mathbf{v}^{\alpha, \alpha\beta} + \hat{e}_{\alpha\beta}^\alpha (E^{\alpha, \alpha\beta} + \frac{1}{2} (v^{\alpha, \alpha\beta})^2)] \\ - [\hat{Q}_{\alpha\beta}^\beta + \hat{\mathbf{T}}_{\alpha\beta}^\beta \cdot \mathbf{v}^{\beta, \alpha\beta} + \hat{e}_{\alpha\beta}^\beta (E^{\beta, \alpha\beta} + \frac{1}{2} (v^{\beta, \alpha\beta})^2)] + \hat{Q}_{s/g}^{\alpha\beta} \end{aligned} \quad (23b)$$

where $\mathbf{t} : \mathbf{d}$ represents, in indicial notation, $t_{ij} d_{ij}$; a comma in the superscript denotes relative values of properties as defined in the last section. Because the contact line possesses no thermodynamic properties, (23b) is subject to

$$\sum_{\alpha\beta} [\hat{Q}_{s/g}^{\alpha\beta} + \hat{\mathbf{S}}_{s/g}^{\alpha\beta} \cdot \mathbf{w}^{\alpha\beta} + \hat{e}_{s/g}^{\alpha\beta} (E^{\alpha\beta} + \frac{1}{2} (w^{\alpha\beta})^2)] = 0 \quad (23c)$$

Balance equations of entropy

For a bulk phase

$$n^\alpha \rho^\alpha \frac{D^\alpha \eta^\alpha}{Dt} - \nabla \cdot (n^\alpha \varphi^\alpha) - n^\alpha \rho^\alpha b^\alpha = \sum_{\beta \neq \alpha} \hat{\Phi}_{\alpha\beta}^\alpha + \Lambda^\alpha \quad (24a)$$

or an interface

$$\begin{aligned} a^{\alpha\beta} \Gamma^{\alpha\beta} \frac{D^{\alpha\beta} \eta^{\alpha\beta}}{Dt} - \nabla \cdot (a^{\alpha\beta} \varphi^{\alpha\beta}) - a^{\alpha\beta} \Gamma^{\alpha\beta} b^{\alpha\beta} \\ = -(\hat{\Phi}_{\alpha\beta}^\alpha + \hat{e}_{\alpha\beta}^\alpha \eta^{\alpha, \alpha\beta}) - (\hat{\Phi}_{\alpha\beta}^\beta + \hat{e}_{\alpha\beta}^\beta \eta^{\beta, \alpha\beta}) + \hat{\Phi}_{s/g}^{\alpha\beta} + \Lambda^{\alpha\beta} \end{aligned} \quad (24b)$$

(24b) is subject to

$$\sum_{\alpha\beta} (\hat{\Phi}_{s/g}^{\alpha\beta} + \hat{c}_{s/g}^{\alpha\beta} \eta^{\alpha\beta}) = 0 \quad (24c)$$

In (24), Λ^α and $\Lambda^{\alpha\beta}$ account for the net production of entropy in α -phase and $\alpha\beta$ -interface, respectively. The second law of thermodynamics requires that³⁵

$$\Lambda = \sum_{\alpha} \Lambda^{\alpha} + \sum_{\alpha\beta} \Lambda^{\alpha\beta} \geq 0 \quad (25)$$

where Λ may be understood as the net production of the entropy of the mixture. Entropy inequality (25) will serve as the thermodynamic restriction on the constitutive equations of unsaturated porous media considered here.

It should be pointed out that balance equations given above are very general in the sense that they take into account the effect of mass exchange between phases, net production of a property within the phases, and external supply of that property to a phase. Also, they include the balance equations for all the interfaces between the bulk phases.

CONSTITUTIVE EQUATIONS, THERMODYNAMIC RESTRICTIONS, AND EQUILIBRIUM CONSIDERATIONS

According to the principle of admissibility, all the variables and the constitutive relations are subject to the second law of thermodynamics, i.e. inequality (25). In the usual way, we define the Helmholtz free energy A for both bulk and interfacial phases by the Legendre transformation:

$$A^\alpha = E^\alpha - \theta^\alpha \eta^\alpha \quad (26a)$$

$$A^{\alpha\beta} = E^{\alpha\beta} - \theta^{\alpha\beta} \eta^{\alpha\beta} \quad (26b)$$

θ^α and $\theta^{\alpha\beta}$ are the temperatures of α -phase and $\alpha\beta$ -phase, respectively. Meanwhile, the entropy fluxes are assumed to be solely due to heat input, and the entropy external source terms are assumed to be only due to external energy source, that is

$$\varphi^\alpha = \mathbf{q}^\alpha / \theta^\alpha, \quad b^\alpha = h^\alpha / \theta^\alpha \quad (27a)$$

$$\varphi^{\alpha\beta} = \mathbf{q}^{\alpha\beta} / \theta^{\alpha\beta}, \quad b^{\alpha\beta} = h^{\alpha\beta} / \theta^{\alpha\beta} \quad (27b)$$

Equations (26) and (27) are used in deriving the entropy inequality (equation (109), Appendix A).

The balance equations introduced in the last section and the entropy inequality (109) suggest that constitutive relations should be prescribed for

$$A^\alpha, A^{\alpha\beta}, n^\alpha \mathbf{t}^\alpha, a^{\alpha\beta} \mathbf{S}^{\alpha\beta}, \hat{\mathbf{T}}_{\alpha\beta}^\alpha, \hat{\mathbf{S}}_{s/g}^{\alpha\beta}, n^\alpha \mathbf{q}^\alpha, \hat{Q}_{\alpha\beta}^\alpha, \hat{Q}_{s/g}^{\alpha\beta}, a^{\alpha\beta} \mathbf{q}^{\alpha\beta}, \eta^\alpha, \eta^{\alpha\beta}, \hat{c}_{\alpha\beta}^\alpha, \hat{c}_{s/g}^{\alpha\beta} \quad (28)$$

which are assumed to be the functions of the following independent variables and their gradients:

$$n^\alpha \rho^\alpha, a^{\alpha\beta} \Gamma^{\alpha\beta}, \mathbf{v}^{\alpha,s}, \mathbf{w}^{\alpha\beta,s}, \mathbf{E}^s, \theta^\alpha, \theta^{\alpha\beta}, \nabla(n^\alpha \rho^\alpha), \nabla(a^{\alpha\beta} \Gamma^{\alpha\beta}), \nabla \theta^\alpha, \nabla \theta^{\alpha\beta} \quad (29)$$

Note that the number of independent variables and the number of mass, linear momentum, and energy equations are the same. Here, we assume that the fluids are inviscid and therefore the

gradients of velocities are excluded in (29). Following the arguments by Drew and Lahey,³⁹ \mathbf{E}^s instead of \mathbf{F}^s and the relative velocities instead of the absolute velocities are chosen as independent constitutive variables so that the requirement of objectivity can be achieved. It is noted that, though the balance equations together with the constitutive relations of the above dependent variables may form a closure problem, it is inadequate for these governing equations to describe the behaviour of a mixture. For instance, in geotechnical problems the change of the porosity with stress variations may be of main concern, but the solution of the aforementioned problem cannot reveal this detail due to the lumping of volume fraction n^α and the stress tensor \mathbf{t}^α . In fact, it had been recognized that intrinsic variables and volume fractions must be introduced independently as constitutive variables to describe the behaviour of individual constituents (phases).⁴⁰ In addition, (3) is a kinematical constraint imposed on the interactions among various phases, implying that the volume fraction n^α and their gradients must be chosen as independent variables. Similarly, specific area $a^{\alpha\beta}$ and their spatial gradients should also be chosen as independent variables. Due to (3a), only two volume fractions are independent. Here, we choose n^ℓ and n^g . Based on these considerations, the independent variables and their gradients may be expressed as

$$\zeta = (\rho^\alpha, \Gamma^{\alpha\beta}, \mathbf{v}^{\alpha,s}, \mathbf{w}^{\alpha\beta,s}, \mathbf{E}^s, \theta^\alpha, \theta^{\alpha\beta}, n^\ell, n^g, a^{\alpha\beta}, \nabla\rho^\alpha, \nabla\Gamma^{\alpha\beta}, \nabla n^\ell, \nabla n^g, \nabla a^{\alpha\beta}, \nabla\theta^\alpha, \nabla\theta^{\alpha\beta}) \quad (30)$$

Because we have introduced five more independent constitutive variables in (30), additional five dependent variables must be chosen to complete the problem. This may be achieved by adding the following variables into the list (28):

$$\dot{n}^\ell, \dot{n}^g, \dot{a}^{s\ell}, \dot{a}^{sg}, \dot{a}^{\ell g} \quad (31)$$

where $\dot{n}^\alpha = D^s n^\alpha / Dt$, etc. The constitutive relations for these five variables are usually called 'closure equations'.⁴¹ These closure equations exist because of the introduction of the volume fraction and area density together with their spatial gradients as independent constitutive variables. Thus far, we obtain the following dependent variables

$$\mathfrak{R} = (A^\alpha, A^{\alpha\beta}, \mathbf{t}^\alpha, \mathbf{S}^{\alpha\beta}, \hat{\mathbf{T}}_{\alpha\beta}^\alpha, \hat{\mathbf{S}}_{s/g}^{\alpha\beta}, \mathbf{q}^\alpha, \hat{\mathbf{Q}}_{\alpha\beta}^\alpha, \hat{\mathbf{Q}}_{s/g}^{\alpha\beta}, \mathbf{q}^{\alpha\beta}, \eta^\alpha, \eta^{\alpha\beta}, \hat{e}_{\alpha\beta}^\alpha, \hat{e}_{s/g}^{\alpha\beta}, \dot{n}^\ell, \dot{n}^g, \dot{a}^{\alpha\beta}) \quad (32)$$

According to the principles of equipresence and determinism,¹⁰ constitutive equations may be expressed, formally, as

$$\mathfrak{R} = \Xi(\zeta) \quad (33)$$

(33) means that each member in (32) is a function of all members in (30). As pointed out by Drew and Lahey,³⁹ it is usually impractical to strictly follow (33), implying that to construct realistic constitutive relations the principle of equipresence must be compromised. As a matter of fact, it is well recognized that^{13,14,15,42} for mixtures consisting of immiscible constituents some variable of λ -phase in (32), e.g. free energy, depend only upon the constitutive variables associated to the λ -phase. Here, we propose that

$$A^\ell = A^\ell(\rho^\ell, \theta^\ell, n^\ell, a^{s\ell}, a^{\ell g}) \quad (34a)$$

$$A^g = A^g(\rho^g, \theta^g, n^g, a^{sg}, a^{\ell g}) \quad (34b)$$

$$A^s = A^s(\rho^s, \theta^s, \mathbf{E}^s, a^{s\ell}, a^{sg}) \quad (34c)$$

$$A^{\alpha\beta} = A^{\alpha\beta}(\Gamma^{\alpha\beta}, \theta^{\alpha\beta}, n^\ell, n^g, a^{\alpha\beta}) \quad \alpha\beta = s\ell, \ell g, sg \quad (34d)$$

The inclusion of area densities in (34a)–(34c) is due to the reason that area densities represent the geometry of the boundary surface of an individual phase. Similar argument is applicable to the inclusion of volume fraction in (34d) for the constitutive functions. The inclusion of the boundary geometry of an individual phase in the constitutive postulates is one of the main distinguishing features of the theory presented in this paper. In (34a)–(34d), we have recovered Curie's principal of symmetry,⁴³ which states that in an isotropic system there is no interaction between two quantities if the difference of the order of their tensor is an odd number. This implies that the inclusion of the velocity and the spatial gradients of n^α , $a^{\alpha\beta}$, and θ in (34) is thermodynamically inadmissible. In (34c), volume fractions are not included, because during deformation of the solid phase its volume fraction is implicitly determined by \mathbf{E}^s and density ρ^s through (16a) if the mass exchange between the solid and the fluids can be neglected.

In Hassanizadeh and Gray's work,^{35,36} porosity n and the degree of saturation S_r have been introduced as independent variables. On the other hand, there are several reasons against these choices. First of all, as previously pointed out, intrinsic properties and the volume fraction have to be used as independent variables in describing the behaviour of an individual constituent. Secondly, n and S_r are not the parameters coming out of the averaging procedure. Their appearance as independent variables will implicitly introduce other assumptions. For example, inclusion of n and S_r implies that n^ℓ and n^g can only influence the behaviour of the mixture through functions $(n^\ell + n^g)$ and $n^\ell/(n^\ell + n^g)$, respectively, and not independently. It has also been argued that⁴⁴ volume fractions instead of n and S_r should be chosen as independent variable in modelling the behaviour of a clay–water colloidal system. Actually, the importance of the inclusion of volume fractions in the constitutive postulates had been recognized by many researchers.^{11,40,42,45,46}

As shown in Appendix A, the admissibility of thermodynamics requires that

$$\eta^\alpha = -\partial A^\alpha / \partial \theta^\alpha \quad (35)$$

$$\eta^{\alpha\beta} = -\partial A^{\alpha\beta} / \partial \theta^{\alpha\beta} \quad (36)$$

$$\mathbf{t}^\ell = -p^\ell \mathbf{I} \quad (37)$$

$$\mathbf{t}^g = -p^g \mathbf{I} \quad (38)$$

$$\mathbf{t}^s = \mathbf{t}^e - p^s \mathbf{I} \quad (39)$$

$$\mathbf{S}^{\alpha\beta} = \gamma^{\alpha\beta} \mathbf{I} \quad (40)$$

where

$$p^\alpha = (\rho^\alpha)^2 \frac{\partial A^\alpha}{\partial \rho^\alpha} \quad (41)$$

$$\gamma^{\alpha\beta} = -(\Gamma^{\alpha\beta})^2 \frac{\partial A^{\alpha\beta}}{\partial \Gamma^{\alpha\beta}} = -a^{\alpha\beta} \Gamma^{\alpha\beta} \frac{\partial A^{\alpha\beta}}{\partial a^{\alpha\beta}} \quad (42)$$

$$\mathbf{t}^e = \rho^s \mathbf{F}^{s\top} \frac{\partial A^s}{\partial \mathbf{E}^s} \mathbf{F}^s \quad (43)$$

Because the fluid and the gas are assumed to be inviscid, the deviatoric parts of the stress tensors in \mathbf{t}^ℓ and \mathbf{t}^g vanish. It should be noted that (35)–(43) are the same as those obtained by

Hassanizadeh and Gray.^{35,36} Among all the terms appearing in (37)–(40), only \mathbf{t}^e is explicitly related to the deformation of the solid phase. As implied to the principle of effective stress, \mathbf{t}^e is related to the effective stress \mathbf{t}' , which has been defined, in soil mechanics, by

$$\mathbf{t}' = n^s \mathbf{t}^e = (1 - n^\ell - n^g) \mathbf{t}^e \quad (44)$$

With (35)–(40) incorporated into (114), the residual entropy inequality becomes

$$\begin{aligned} \Lambda &= \Lambda(i^z, \dot{a}^{\alpha\beta}, v^{\alpha,s}, w^{\alpha\beta,s}, \nabla\theta^\alpha, \nabla\theta^{\alpha\beta}, \theta^{\alpha\beta,s}, \theta^{\alpha\beta,\alpha}, \hat{e}_{\alpha\beta}^z, \hat{e}_{s\ell}^{\alpha\beta}) \\ &= \Lambda(z_1, z_2, \dots, z_9, z_{10}) \geq 0 \quad \alpha = \ell, g \quad \alpha\beta = s\ell, \ell g, sg \end{aligned} \quad (45)$$

To examine the proposed constitutive relations further, we shall consider appropriate thermodynamic equilibrium states for the material. Thermodynamic equilibrium is achieved if and only if all the following conditions are valid:

$$1. \quad z_1 = z_2 = \dots = z_{10} = 0 \quad (46a)$$

$$2. \quad (\partial\Lambda/\partial z_i)|_{z_i=0} = 0 \quad (46b)$$

$$3. \quad \|\partial^2\Lambda/\partial z_i \partial z_j\| \geq 0 \quad (46c)$$

Restrictions (46a) and (46b) yield

$$(p_{\ell s})_{\text{eq.}} = p^\ell - p^s \quad (47)$$

$$(p_{gs})_{\text{eq.}} = p^g - p^s \quad (48)$$

$$(\hat{\mathbf{T}}_{\ell s}^\ell + \hat{\mathbf{T}}_{\ell g}^\ell)_{\text{eq.}} = \left(p^\ell - n^\ell \rho^\ell \frac{\partial A^\ell}{\partial n^\ell} \right) \nabla n^\ell - \sum_{\beta \neq \ell} n^\ell \rho^\ell \frac{\partial A^\ell}{\partial a^{\ell\beta}} \nabla a^{\ell\beta} \quad (49a)$$

$$(\hat{\mathbf{T}}_{gs}^g + \hat{\mathbf{T}}_{g\ell}^g)_{\text{eq.}} = \left(p^g - n^g \rho^g \frac{\partial A^g}{\partial n^g} \right) \nabla n^g - \sum_{\beta \neq g} n^g \rho^g \frac{\partial A^g}{\partial a^{g\beta}} \nabla a^{g\beta} \quad (49b)$$

$$(\hat{\mathbf{S}}_{s\ell}^{\alpha\beta} - \hat{\mathbf{T}}_{\alpha\beta}^z - \hat{\mathbf{T}}_{\alpha\beta}^g)_{\text{eq.}} = -a^{\alpha\beta} \Gamma^{\alpha\beta} \frac{\partial A^{\alpha\beta}}{\partial n^\ell} \nabla n^\ell - a^{\alpha\beta} \Gamma^{\alpha\beta} \frac{\partial A^{\alpha\beta}}{\partial n^g} \nabla n^g \quad (50)$$

$$(\mathbf{q}^z)_{\text{eq.}} = (\mathbf{q}^{\alpha\beta})_{\text{eq.}} = 0 \quad (51)$$

$$\left(n^z \rho^z \frac{\partial A^z}{\partial a^{\alpha\beta}} + n^\beta \rho^\beta \frac{\partial A^\beta}{\partial a^{\alpha\beta}} \right)_{\text{eq.}} = 0 \quad (52)$$

$$(\hat{Q}_{\alpha\beta}^z)_{\text{eq.}} = 0 \quad (53a)$$

$$(\hat{Q}_{s\ell}^{\alpha\beta})_{\text{eq.}} = 0 \quad (53b)$$

$$(G^{\alpha\beta,\alpha})_{\text{eq.}} = 0 \quad (54)$$

where $(\)_{\text{eq.}}$ represents an equilibrium state; $p_{\ell s}$ and p_{gs} are capillary pressures defined by

$$\frac{p_{\ell s}}{\theta^\ell} = \frac{n^\ell \rho^\ell}{\theta^\ell} \frac{\partial A^\ell}{\partial n^\ell} + \sum_{\alpha\beta} \frac{a^{\alpha\beta} \Gamma^{\alpha\beta}}{\theta^{\alpha\beta}} \frac{\partial A^{\alpha\beta}}{\partial n^\ell} \quad (55a)$$

$$\frac{p_{gs}}{\theta^g} = \frac{n^g \rho^g}{\theta^g} \frac{\partial A^g}{\partial n^g} + \sum_{\alpha\beta} \frac{a^{\alpha\beta} \Gamma^{\alpha\beta}}{\theta^{\alpha\beta}} \frac{\partial A^{\alpha\beta}}{\partial n^g} \quad (55b)$$

In the derivation of (50) and (52), definition (42) has been utilized. Equation (47) has been previously derived and used to explain the expansive behaviour of a clay–water system.^{4,7} However, since Achanta *et al.*^{4,7} did not consider A^g to be a function of n^g they were unable to derive (48).

FLOW EQUATIONS FOR THE LIQUID AND THE GAS

It should be noted that (47)–(54) are valid only at equilibrium state. However, the equilibrium restrictions are too strong for many practical problems. For instance, when sand is subject to a dynamic load, the relative velocity between the phases may not be trivial. In this section, we will develop the generalized Darcy's laws for liquid and gas. We assume that isothermal conditions prevails in the mixture so that $\theta^\alpha = \theta^{\alpha\beta} = \text{constant}$, and the exchange terms of properties through the contact line can be neglected. Although the above assumptions are not necessary for the following derivations, it simplifies the derivations significantly. From (47)–(54), it is understood that all the terms in the l.h.s. of these equations consist of two parts: one is equilibrium part as expressed by these equations; the other is dissipative part, which will be zero at equilibrium state. Accordingly, the entropy also has two parts. Following Eringen¹⁴ and using the entropy inequality (114) (Appendix A), we have the following residual dissipative entropy inequality, namely

$$\begin{aligned} \theta \Lambda = & \sum_{\alpha \neq s} \dot{n}^\alpha \bar{p}_{zs} - \sum_{\alpha\beta} \dot{a}^{\alpha\beta} \left(n^\alpha \rho^\alpha \frac{\partial A^\alpha}{\partial a^{\alpha\beta}} + n^\beta \rho^\beta \frac{\partial A^\beta}{\partial a^{\alpha\beta}} \right) \\ & - \sum_{\alpha \neq s} \mathbf{v}^{\alpha,s} \cdot \bar{\tau}^\alpha - \sum_{\alpha\beta} \mathbf{w}^{\alpha\beta,s} \cdot \bar{\tau}^{\alpha\beta} \\ & + \sum_{\alpha} \sum_{\beta \neq \alpha} \hat{e}_{\alpha\beta}^\alpha [G^{\alpha\beta,\alpha} - \frac{1}{2} (W^{\alpha\beta,\alpha})^2] \\ & \geq 0 \end{aligned} \quad (56)$$

where \bar{p}_{zs} is the dissipative part of capillary pressure p_{zs} ; $\bar{\tau}^\alpha$ and $\bar{\tau}^{\alpha\beta}$ are dissipative parts of the exchanges of linear momentum. Therefore,

$$p_{zs} = (p_{zs})_{\text{eq.}} + \bar{p}_{zs} \quad \alpha = \ell, g \quad (57)$$

$$\sum_{\beta \neq \alpha} \hat{\mathbf{T}}_{\alpha\beta}^\alpha = \left(\sum_{\beta \neq \alpha} \hat{\mathbf{T}}_{\alpha\beta}^\alpha \right)_{\text{eq.}} + \bar{\tau}^\alpha \quad \alpha = \ell, g \quad (58)$$

and

$$\hat{\mathbf{S}}_{s/g}^{\alpha\beta} - \hat{\mathbf{T}}_{\alpha\beta}^\alpha - \hat{\mathbf{T}}_{\alpha\beta}^\beta = (\hat{\mathbf{S}}_{s/g}^{\alpha\beta} - \hat{\mathbf{T}}_{\alpha\beta}^\alpha - \hat{\mathbf{T}}_{\alpha\beta}^\beta)_{\text{eq.}} + \bar{\tau}^{\alpha\beta} \quad (59)$$

The first terms of the r.h.s. in (57)–(59) are the equilibrium parts given by (47) and (48), (49) and (50), respectively. Assume the deviation from an equilibrium state is small so that the dissipative

drag forces $\bar{\tau}^\alpha$ and $\bar{\tau}^{\alpha\beta}$ may be expressed as linear functions of relative velocities, therefore

$$\bar{\tau}^\ell = -\boldsymbol{\mu}_\ell^\ell \cdot \mathbf{v}^{\ell,s} - \boldsymbol{\mu}_g^\ell \cdot \mathbf{v}^{g,s} - \sum_{\alpha\beta} \boldsymbol{\mu}_{\alpha\beta}^\ell \cdot \mathbf{w}^{\alpha\beta,s} \quad (60)$$

$$\bar{\tau}^g = -\boldsymbol{\mu}_\ell^g \cdot \mathbf{v}^{\ell,s} - \boldsymbol{\mu}_g^g \cdot \mathbf{v}^{g,s} - \sum_{\alpha\beta} \boldsymbol{\mu}_{\alpha\beta}^g \cdot \mathbf{w}^{\alpha\beta,s} \quad (61)$$

and

$$\bar{\tau}^{\gamma\delta} = -\boldsymbol{\mu}_\ell^{\gamma\delta} \cdot \mathbf{v}^{\ell,s} - \boldsymbol{\mu}_g^{\gamma\delta} \cdot \mathbf{v}^{g,s} - \sum_{\alpha\beta} \boldsymbol{\mu}_{\alpha\beta}^{\gamma\delta} \cdot \mathbf{w}^{\alpha\beta,s} \quad (62)$$

where $\alpha\beta$ and $\gamma\delta = s\ell, \ell g, sg$; $\boldsymbol{\mu}_\alpha^\beta$ and $\boldsymbol{\mu}_{\alpha\beta}^{\gamma\delta}$ are material coefficients. It is noted that (60)–(62) have included coupling effects between the phases, and $\boldsymbol{\mu}_\alpha^\beta$ ($\alpha \neq \beta$), $\boldsymbol{\mu}_{\alpha\beta}^\gamma$, $\boldsymbol{\mu}_\alpha^{\gamma\delta}$ and $\boldsymbol{\mu}_{\alpha\beta}^{\gamma\delta}$ ($\alpha\beta \neq \gamma\delta$) are the coupling material coefficients. Under the restriction (46c), it can be shown that

$$\boldsymbol{\mu}_\alpha^\beta + (\boldsymbol{\mu}_\beta^\alpha)^T, \quad \boldsymbol{\mu}_{\alpha\beta}^\gamma + (\boldsymbol{\mu}_\gamma^{\alpha\beta})^T, \quad \boldsymbol{\mu}_\alpha^{\gamma\delta} + (\boldsymbol{\mu}_{\alpha\beta}^{\gamma\delta})^T \quad (63)$$

are positive and semi-definite, where $\alpha, \beta, \gamma = \ell, g$; $\alpha\beta, \gamma\delta = s\ell, sg, \ell g$.

Equations (60)–(63) may be utilized to derive the balance equation of momentum for fluid and gas as well as interfaces. Substitution of (58) into (17a) and incorporation of (60) and (61) yield

$$\begin{aligned} \nabla p^\ell - \rho^\ell \mathbf{g} + \rho^\ell \frac{D^\ell \mathbf{v}^\ell}{Dt} = & -\rho^\ell \left(\frac{\partial A^\ell}{\partial n^\ell} \nabla n^\ell + \sum_{\beta \neq \ell} \frac{\partial A^\ell}{\partial a^{\ell\beta}} \nabla a^{\ell\beta} \right) \\ & - \frac{1}{n^\ell} \left(\sum_{\beta \neq s} \boldsymbol{\mu}_\beta^\ell \cdot \mathbf{v}^{\beta,s} + \sum_{\beta\gamma} \boldsymbol{\mu}_{\beta\gamma}^\ell \cdot \mathbf{w}^{\beta\gamma,s} \right) \end{aligned} \quad (64a)$$

and

$$\begin{aligned} \nabla p^g - \rho^g \mathbf{g} + \rho^g \frac{D^g \mathbf{v}^g}{Dt} = & -\rho^g \left(\frac{\partial A^g}{\partial n^g} \nabla n^g + \sum_{\beta \neq g} \frac{\partial A^g}{\partial a^{g\beta}} \nabla a^{g\beta} \right) \\ & - \frac{1}{n^g} \left(\sum_{\beta \neq s} \boldsymbol{\mu}_\beta^g \cdot \mathbf{v}^{\beta,s} + \sum_{\beta\gamma} \boldsymbol{\mu}_{\beta\gamma}^g \cdot \mathbf{w}^{\beta\gamma,s} \right) \end{aligned} \quad (64b)$$

Utilizing constitutive assumptions (34a) and (34b) and assuming that the gradients of intrinsic densities are trivial, we can prove that (64a) and (64b) can be written in a more composite form, i.e.

$$\rho^\alpha \nabla G^\alpha - \rho^\alpha \mathbf{g} + \rho^\alpha \frac{D^\alpha \mathbf{v}^\alpha}{Dt} = -\frac{1}{n^\alpha} \left(\sum_{\beta \neq s} \boldsymbol{\mu}_\beta^\alpha \cdot \mathbf{v}^{\beta,s} + \sum_{\beta\gamma} \boldsymbol{\mu}_{\beta\gamma}^\alpha \cdot \mathbf{v}^{\beta\gamma,s} \right) \quad \alpha = \ell, g \quad (65a)$$

where G^α is the Gibbs' thermodynamic potential of α -phase defined by

$$G^\alpha = A^\alpha + p^\alpha / \rho^\alpha \quad (65b)$$

(65a) is referred to as the generalized Darcy's law, which represents the most general form of flow equation for the fluids. This result is consistent with Bowen's work.¹¹ However, we have explicitly included the interfacial effect. It can be concluded from (65a) that the Gibbs' thermodynamic potential is the driving potential for flow rather than the pressure. This is true, especially for

unsaturated systems, where use of only pressure gradient and the gravity is not sufficient to describe the transport process of the fluids.⁴⁸ Indeed, the flow of a fluid phase is determined not only by its own properties, such as pressure, density and viscosity, but also by its environmental and boundary properties, such as temperature, specific area, and surface tension. This implies that a unique relationship between the gradient of pressure and the fluid velocity, as indicated by the classic Darcy's law, does not exist in a general sense.

If the soil is saturated, (64a) becomes

$$\nabla p^\ell - \rho^\ell \mathbf{g} + \rho^\ell \frac{D^\ell \mathbf{v}^\ell}{Dt} = -\rho^\ell \frac{\partial A^\ell}{\partial n^\ell} \nabla n^\ell - \frac{1}{n^\ell} \boldsymbol{\mu}_\ell^\ell \cdot \mathbf{v}^\ell, s \quad (66)$$

Because for saturated soils the free energy of the liquid is independent of the porosity, the first term in the r.h.s of (66) will vanish and (66) becomes the traditional balance equation of momentum for the liquid in a saturated soil and had been used in the dynamic analysis of saturated porous media.³⁻⁶ This equation can be simplified into the standard form of Darcy's law by neglecting inertial effects.

Similarly, substitution of (59) into (17b) and incorporation of (62) yield

$$\nabla G^{\alpha\beta} - \Gamma^{\alpha\beta} \mathbf{g} + \Gamma^{\alpha\beta} \frac{D^{\alpha\beta} \mathbf{w}^{\alpha\beta}}{Dt} = -\frac{1}{a^{\alpha\beta}} \left(\sum_{\gamma \neq s} \boldsymbol{\mu}_\gamma^{\alpha\beta} \cdot \mathbf{v}^{\gamma, s} + \sum_{\gamma \delta} \boldsymbol{\mu}_{\gamma\delta}^{\alpha\beta} \cdot \mathbf{w}^{\gamma\delta, s} \right) \quad (67)$$

where $G^{\alpha\beta}$ is the chemical potential of $\alpha\beta$ -interface and defined by

$$G^{\alpha\beta} = A^{\alpha\beta} - \gamma^{\alpha\beta} / \Gamma^{\alpha\beta} \quad (68)$$

Darcy's law (65) and the flow equations for the interface (67) are very general in the sense that the inertial and interaction terms are included. However, they provide a rational basis on which governing equations for fluid flow within an unsaturated soil can be derived.

THERMODYNAMIC BASIS FOR THE USE OF TWO STRESS STATE VARIABLES

The principle of effective stress⁴⁹ has been well known in geotechnical engineering, concrete technology, geophysics and other disciplines. It plays a key role in the development of soil mechanics. In this section, the validity of the principle of effective stress in unsaturated soil mechanics will be examined, and its limitations in applications will be discussed.

According to Terzaghi,⁴⁹ the principle of effective stress may be stated in the form of two propositions:

- (i) all measurable effects of a change of stress, such as compression, distortion, and a change of shear strength of a soil are exclusively due to change in effective stress,
- (ii) the effective stress is defined as the excess of the total applied stress over the pore pressure.

From these two propositions, we understand that effective stress formulation basically is a constitutive relation, which had been recognized by Biot¹ and Zienkiewicz.⁵⁰ It is because of proposition (ii) that the effective stress concept has been very effectively used in the saturated soil mechanics. However, its extension to unsaturated soils is not straightforward. Bishop²⁹ suggested an intuitive formulation of the effective stress for unsaturated soils:

$$\sigma' = (\sigma - u_a) + \chi(u_a - u_w) \quad (69)$$

where χ is assumed to be the function of the degree of saturation. Equation (69) has been subject to widespread study. Thus far, sufficient experimental data show that χ is a function not only of S_r , but also of soil type and stress path.³⁰ This observation implies that the application of this effective stress formula in unsaturated soil problems is very limited.

To discuss this point further, we resort to (47) and (48). From these equations, it can be seen that among p^ℓ , p^g and p^s there exist a balance, and the differences among these pressures are called capillary pressures, one of which, known as suction in practice, is defined by the difference between the gas pressure and the liquid pressure. This balance may be destroyed during the deformation of the solid phase until a new equilibrium state is achieved. It is noted that the capillary pressures discussed here have explicitly included the interfacial effect (see equation (55)). As previously mentioned, stress tensor of the solid phase \mathbf{t}^s consists of two part: the first part, called intergranular force \mathbf{t}^e ,⁴ is a function of solid deformation, by which effective stress \mathbf{t}' is defined as $\mathbf{t}' = n^s \mathbf{t}^e$; the second part of \mathbf{t}^s , designated by p^s , is independent of deformation \mathbf{E}^s and is compensated by the pressures of the liquid and the gas as well as the surface tensions of the interfaces.

Here, we define the suction as

$$S_c = n^g \rho^g \frac{\partial A^g}{\partial n^g} + \sum_{\alpha\beta} a^{\alpha\beta} \Gamma^{\alpha\beta} \frac{\partial A^{\alpha\beta}}{\partial n^g} - \left(n^\ell \rho^\ell \frac{\partial A^\ell}{\partial n^\ell} + \sum_{\alpha\beta} a^{\alpha\beta} \Gamma^{\alpha\beta} \frac{\partial A^{\alpha\beta}}{\partial n^\ell} \right) \quad (70)$$

Clearly, at equilibrium state

$$S_c = p^g - p^\ell \quad (71)$$

Assume that there exists a function ϕ such that

$$\phi S_c = (p_{gs})_{eq.} = n^g \rho^g \frac{\partial A^g}{\partial n^g} + \sum_{\alpha\beta} a^{\alpha\beta} \Gamma^{\alpha\beta} \frac{\partial A^{\alpha\beta}}{\partial n^g} \quad (72)$$

where S_c has been defined by (70). Obviously, if S_c is not equal to zero, the soil is unsaturated, and function ϕ always exists. If the soil is saturated, S_c and the r.h.s. of (72) will vanish at equilibrium, and ϕ can be set to unity. Using (72), (48) can be written as

$$p^g - p^s = \phi S_c \quad (73)$$

where

$$\phi = \phi(n^\ell, \rho^\ell, n^g, \rho^g, a^{\alpha\beta}, \Gamma^{\alpha\beta}, \theta) \quad (74)$$

By incorporating (71), (73) may be written as

$$p^s = \phi p^\ell + (1 - \phi) p^g \quad (75)$$

(74) implies that generally $\phi \neq S_r$. It is reasonable to argue that because of the existence of interfaces part of p^s should be compensated by the surface tension. Due to the dependence of ϕ on the interfacial properties $a^{\alpha\beta}$ and $\Gamma^{\alpha\beta}$, the r.h.s. of (75) does include the interfacial effects. Substituting (37)–(40) into (19) and incorporating (75) yields

$$\mathbf{t}' = \mathbf{t} + p^g \mathbf{I} - \kappa(p^g - p^\ell) \mathbf{I} + \left(\sum_{\alpha\beta} a^{\alpha\beta} \gamma^{\alpha\beta} \right) \mathbf{I} \quad (76)$$

where

$$\kappa = (1 - n^g)\phi + n^\ell(1 - \phi) \quad (77)$$

Thus κ is a function of n^ℓ , ρ^ℓ , n^g , ρ^g , $a^{\alpha\beta}$, $\Gamma^{\alpha\beta}$, and θ ; \mathbf{t} and \mathbf{t}' are the total and effective stress tensors, respectively. It is interesting to note that Matyas and Radhakrishna⁵¹ obtained an effective stress (called intergranular stress in that paper) formulation by the phenomenological method, that is

$$\mathbf{t}' = \mathbf{t} + p^g \mathbf{I} - A_w(p^g - p^\ell) \mathbf{I} + \int \gamma'^g dx \mathbf{I} \quad (78)$$

where A_w is areal fraction of liquid on which stress is defined; γ'^g is the surface tension force per unit length of gas-liquid meniscus. Recently, Pietruszczak and Pande⁵² have also intuitively derived a similar formulation of effective stress for unsaturated soils with relatively high saturation ($S_r \geq 70\%$):

$$\mathbf{t}' = \mathbf{t} + [S_r p^\ell + (1 - S_r) p^g] \mathbf{I} + \frac{2}{3} \gamma'^g \frac{\sqrt{1 - S_r}}{\rho_v} \mathbf{I} \quad (79)$$

where ρ_v is called 'averaged pore size', defined as the ratio of total void volume to internal solid surface area. The comparison of (78) and (79) with (76) shows that (78) and (79) are special cases of (76), on the other hand, phenomenological procedure cannot rigorously capture the effects of interfaces. As shown in (76), the effective stress expression possesses a complex nature, which is induced by the interactions among various phases.

In practice, the effects of $\gamma^{s\ell}$ and γ^{sg} may be neglected. Thus the last term in (76) becomes

$$\sum_{\alpha\beta} a^{\alpha\beta} \gamma^{\alpha\beta} = a^{\ell g} \gamma'^g \quad (80)$$

As will be proved in the next section, there exists a function v such that

$$\gamma'^g = v(p^g - p^\ell) \quad (81)$$

where

$$v = \frac{\partial n^g / \partial x_i}{\partial a^{\ell g} / \partial x_i} \quad i = 1 \text{ or } 2 \text{ or } 3 \quad (82)$$

Substituting (80) and (81) into (76), one may obtain a general effective stress formulation

$$\mathbf{t}' = \mathbf{t} + p^g \mathbf{I} - \chi(p^g - p^\ell) \mathbf{I} \quad (83)$$

where

$$\chi = \kappa + v a^{\ell g} \quad (84)$$

(83) takes exactly the same form as Bishop's formula. It is known that when the soil becomes saturated $\chi = 1$ and (83) reduces to Terzaghi's effective stress formula. Equation (84) indicates that the parameter χ depends on n^ℓ , n^g , $a^{\ell g}$, ρ^ℓ , ρ^g , $\Gamma^{\ell g}$, and

$$\frac{\partial n^g / \partial x_i}{\partial a^{\ell g} / \partial x_i},$$

implying that effective stress is stress path and stress history dependent, and may vary between different types of soils. As will be discussed later, the inclusion of the ratio of spatial gradients of n^g and a'^g implies that the geometry and the curvature of boundary surfaces of individual phases significantly influence the interactions among various phases. This observation is important in constructing constitutive relations. From (83), it can be seen that the calculation of effective stress involves great uncertainties, especially for the soils with relatively low degree of saturation, where the interfacial effect is strong. This makes it impossible to utilize a single effective stress variable to capture the behaviour of unsaturated soils. If net stress tensor ($\mathbf{t} + p^g \mathbf{I}$) and suction $(p^g - p') \mathbf{I}$ are introduced as independent stress state variables, the calculation of effective stress becomes the evaluation of the constitutive relation of the solid skeleton.

Substitution of (83) into (44) leads to

$$\mathbf{t} + p^g \mathbf{I} - \chi(p^g - p') \mathbf{I} = (1 - n' - n^g) \rho^s \mathbf{F}^{s^T} \frac{\partial A^s}{\partial \mathbf{E}^s} \mathbf{F}^s \quad (85)$$

Because n' , n^g , a'^g , ρ' , ρ^g , Γ'^g , and

$$\frac{\partial n^g / \partial x_i}{\partial a'^g / \partial x_i}$$

may change intensively during deformation of solid skeleton, the behaviour of the material represented by (85) basically is non-linear. As discussed previously, phenomenological approach seem to be the best way to describe the non-linear stress-strain behaviour of the solid skeleton. An elastoplastic constitutive model for unsaturated soils developed within the framework of bounding surface plasticity and critical state soil mechanics principle was presented by Muraleetharan and Nedunuri.³² The bounding surface was described in a four-dimensional space consisting of three net stress invariants and the matric suction. Plastic void ratio and plastic specific water volume (volume of liquid and solids in an element of soil containing unit volume of solids) were used as internal variables. Isotropic hardening, associative flow rule and radial mapping were used to calculate the plastic strains. This constitutive model can be described by

$$\dot{\boldsymbol{\sigma}}^N = \mathbf{D} : \dot{\mathbf{E}}^s \quad (86)$$

where $\dot{\boldsymbol{\sigma}}^N$ is the rate of net stress tensor and \mathbf{D} is the elastoplastic moduli tensor.

THE GOVERNING EQUATIONS FOR DYNAMIC PROBLEMS

In this section, we will elaborate on the field equations governing the behaviour of unsaturated soils under dynamic loading and isothermal conditions. Guided by our experience with dynamic analyses of saturated soils⁶ and seeking a set of equations which can be implemented into techniques such as finite element methods, we will make certain assumptions. These assumptions, however, are realistic and do not compromise much of the generality. The assumptions are:

- (i) the mass exchange between the solid and the liquid or the gas can be neglected,
- (ii) the interfaces between the solid and the liquid or the gas have no thermodynamic properties; the only considered surface tension is the one on the contractile skin between the liquid and the gas,
- (iii) contact line has no thermodynamic properties, and exchange terms associated with the contact line can be neglected,

- (iv) the mass, the inertial effect and the body force of an interface can be neglected; the only considered property of an interface is the surface tension,
- (v) the multiplication of a mass exchange term by a relative velocity is a high-order small quantity and may be neglected,
- (vi) gravity is the only body force.

Under the restriction (i), (iii) and (iv), the mass balance (16) reduces to

$$\frac{D^s(n^s\rho^s)}{Dt} + n^s\rho^s(\nabla \cdot \mathbf{v}^s) = 0 \quad (87a)$$

$$\frac{D^\ell(n^\ell\rho^\ell)}{Dt} + n^\ell\rho^\ell(\nabla \cdot \mathbf{v}^\ell) = \hat{e}_{\ell g}^\ell \quad (87b)$$

$$\frac{D^g(n^g\rho^g)}{Dt} + n^g\rho^g(\nabla \cdot \mathbf{v}^g) = \hat{e}_{\ell g}^g \quad (87c)$$

$$-\hat{e}_{\ell g}^\ell - \hat{e}_{\ell g}^g = 0 \quad (87d)$$

With restrictions (ii)–(vi), (17) becomes

$$n^s\rho^s \frac{D^s\mathbf{v}^s}{Dt} - \nabla \cdot (n^s\mathbf{t}^s) - n^s\rho^s\mathbf{g} = \hat{\mathbf{T}}_{s\ell}^s + \hat{\mathbf{T}}_{sg}^s \quad (88a)$$

$$n^\ell\rho^\ell \frac{D^\ell\mathbf{v}^\ell}{Dt} - \nabla \cdot (n^\ell\mathbf{t}^\ell) - n^\ell\rho^\ell\mathbf{g} = \hat{\mathbf{T}}_{\ell s}^\ell + \hat{\mathbf{T}}_{\ell g}^\ell \quad (88b)$$

$$n^g\rho^g \frac{D^g\mathbf{v}^g}{Dt} - \nabla \cdot (n^g\mathbf{t}^g) - n^g\rho^g\mathbf{g} = \hat{\mathbf{T}}_{\ell g}^g + \hat{\mathbf{T}}_{sg}^g \quad (88c)$$

and

$$-\nabla \cdot (a^{\ell g}\gamma^{\ell g}\mathbf{I}) = -\hat{\mathbf{T}}_{\ell g}^\ell - \hat{\mathbf{T}}_{\ell g}^g \quad (89a)$$

$$0 = -\hat{\mathbf{T}}_{sg}^s - \hat{\mathbf{T}}_{sg}^g \quad (89b)$$

$$0 = -\hat{\mathbf{T}}_{s\ell}^s - \hat{\mathbf{T}}_{s\ell}^\ell \quad (89c)$$

The total balance equation of the mixture (18) can be written as

$$n^s\rho^s \frac{D^s\mathbf{v}^s}{Dt} + n^\ell\rho^\ell \frac{D^\ell\mathbf{v}^\ell}{Dt} + n^g\rho^g \frac{D^g\mathbf{v}^g}{Dt} - \nabla \cdot \mathbf{t} - \rho\mathbf{g} = 0 \quad (90)$$

where

$$\rho = \sum_{\alpha} n^{\alpha}\rho^{\alpha} + \sum_{\alpha\beta} a^{\alpha\beta}\Gamma^{\alpha\beta} = \sum_{\alpha} n^{\alpha}\rho^{\alpha} \quad (91)$$

$$\mathbf{t} = \sum_{\alpha} n^{\alpha}\mathbf{t}^{\alpha} + a^{\ell g}\gamma^{\ell g}\mathbf{I} \quad (92)$$

Equations (87a)–(87c), (88) and (90) consists of 7 balance equations, among which only six are independent. Usually, the overall balance equation of the linear momentum for the mixture, i.e. (90), will be chosen as an independent equation instead of (88a).

Equation (65a) gives the flow equations for the flow of fluids. Generally, it is necessary to develop constitutive relationships for the Gibbs' thermodynamic potentials G^ℓ and G^g . At this stage, however, we may introduce an approach prevailing in practice, that is, consider the gradient of Helmholtz's free energy, A^α ($\alpha = \ell, g$), implicitly by assuming μ_ℓ' and μ_g^g to be the functions of the degree of saturation and neglecting the coupling terms, so that (65a) leads to

$$\nabla p^\ell - \rho^\ell \mathbf{g} + \rho^\ell \frac{D^\ell \mathbf{v}^\ell}{Dt} = -\mu_\ell' \cdot \mathbf{v}^{\ell,s} \quad (93a)$$

$$\nabla p^g - \rho^g \mathbf{g} + \rho^g \frac{D^g \mathbf{v}^g}{Dt} = -\mu_g^g \cdot \mathbf{v}^{g,s} \quad (93b)$$

How well this approach can capture the essential features of flow within an unsaturated system is yet to be investigated.

Similarly, (67) may serve as the balance equations for the interfaces. Traditionally, the importance of these equations has been overlooked. As discussed in previous sections, the linear momentum balances of the interfaces intrinsically account for the interactions among various bulk phases. For our purpose, the properties of interfaces are not of main concern. However, equation (67) is a restriction to the interactions among various bulk phases. Due to the tremendous difficulty involved in applying (67), we have to find equivalent relationships to account for this restriction. We start with the definition of the momentum exchange term $\hat{\mathbf{T}}_{\alpha\beta}^\alpha$, i.e.³⁵

$$\hat{\mathbf{T}}_{\alpha\beta}^\alpha = \frac{1}{\delta V} \int_{\delta A_{\alpha\beta}} \mathbf{n}^\alpha \cdot [\mathbf{t}|_\alpha + \rho|_\alpha \tilde{\mathbf{v}}^\alpha (\mathbf{w}|_{\alpha\beta} - \mathbf{v}|_\alpha)] da \quad (94)$$

where δV is the volume of REV; $\delta A_{\alpha\beta}$ is the surface area of $\alpha\beta$ -interface within REV; \mathbf{n}^α is unit normal vector to $\delta A_{\alpha\beta}$ and pointing out of α -phase; the terms with $|_\alpha$ represent the microscopic properties of α -phase; $\tilde{\mathbf{v}}^\alpha (= \mathbf{v}|_\alpha - \mathbf{v}^\alpha)$ is the diffusion velocity of α -phase. The second term within the square brackets of the r.h.s. accounts for the momentum exchange due to rate of mass exchange between bulk phase and interface. If the rate of mass exchange is small this term may be neglected as that

$$\hat{\mathbf{T}}_{\alpha\beta}^\alpha = \frac{1}{\delta V} \int_{\delta A_{\alpha\beta}} \mathbf{n}^\alpha \cdot \mathbf{t}|_\alpha da \quad (95)$$

Summation of (89a)–(89c) yields

$$-\nabla \cdot (\mathbf{a}^{\ell g} \gamma^{\ell g} \mathbf{I}) = -\sum_\alpha \sum_{\beta \neq \alpha} \hat{\mathbf{T}}_{\alpha\beta}^\alpha = -\sum_\alpha \frac{1}{\delta V} \sum_{\beta \neq \alpha} \int_{\delta A_{\alpha\beta}} \mathbf{n}^\alpha \cdot \mathbf{t}|_\alpha da \quad (96)$$

It has been shown that⁵³

$$-\frac{1}{\delta V} \sum_{\beta \neq \alpha} \int_{\delta A_{\alpha\beta}} \mathbf{n}^\alpha da = \nabla n^\alpha \quad (97)$$

Assume that the variation of $\mathbf{t}|_\alpha$ over boundary $\delta A_{\alpha\beta}$ is insignificant and the distribution of the porosity is locally uniform, then

$$-\sum_{\alpha} \frac{1}{\delta V} \sum_{\beta \neq \alpha} \int_{\delta A_{\alpha\beta}} \mathbf{n}^\alpha \cdot \mathbf{t}|_\alpha d\mathbf{a} = \sum_{\alpha} \nabla n^\alpha \cdot \mathbf{t}^\alpha = -(p^g - p^\ell) \nabla n^g \quad (98)$$

Substituting (98) into (96) and manipulating the resultant, we obtain

$$p^g - p^\ell = \frac{\partial(a'^g \gamma'^g)/\partial x_i}{\partial n^g/\partial x_i} = \frac{\partial a'^g/\partial x_i}{\partial n^g/\partial x_i} \gamma'^g, \quad i = 1 \text{ or } 2 \text{ or } 3 \quad (99)$$

In the derivation of the second equation in (99), we assume that the capillary equilibrium state prevails in the mixture so that the gradient of surface tension, i.e. $\partial \gamma'^g/\partial x_i$, is zero.⁵⁴ For an isothermal problem, the capillary equilibrium state can be easily achieved. Equation (99) represents the relationship between the suction and the macroscopic surface tension. Traditionally, it has been considered as a closure equation.⁴¹ On the other hand, as is just shown, it can be directly derived from the overall momentum balance equations of the interfaces. This result is very important for correctly constructing governing equations of an unsaturated soil problem.

Equation (99) shows that, when the soil deforms under isothermal condition, the change of the suction is controlled by the ratio of the gradients $\partial a'^g/\partial x_i$ and $\partial n^g/\partial x_i$ (equal in all directions). In soil science, the deformation is not of main concern, and hence the suction may be assumed to be a function of the degree of saturation S_r . On the other hand, in a soil mechanics-oriented problem, the soil skeleton will deform and the suction is spatially changeable. The functional dependence of suction on a single parameter S_r by no means can capture the main feature of the suction.

We propose that (99) be changed to

$$p^g - p^\ell = \frac{\delta}{\gamma_v} \gamma'^g \quad (100)$$

where γ_v is the equivalent average radius of air bubbles, which is a function of volume fraction n^g and area density a'^g ; δ is a parameter called 'the coefficient of curvature variation' of the contractile skin, which depends on pore geometry, porosity, and the degree of saturation. When uniform sands have relatively high saturation, the air bubbles may exist as spheres entrapped in water and

$$\frac{\partial a'^g/\partial x_i}{\partial n^g/\partial x_i},$$

can be calculated to be equal to $2/r$, where r is the radius of air bubbles. In this case, (99) is exactly the same as Young–Laplace equation. If $\gamma_v = r$, (100) implies that $\delta = 2$. Generally, δ is subject to change whenever water content changes or the soil deforms, i.e. the geometric configuration of the pore changes. It should be pointed out that because the soil subjected to drying and wetting at the same degree of saturation will have different meniscus of the contractile skin, (99) and (100) basically can capture the hysteresis behaviour in the soil–water relationship.

In order to incorporate (99) or (100) into the dynamics analysis, the suction must be expressed as the function of other easily measurable variables. Based on the discussions above, we propose that

$$p^g - p^\ell = h_c(\text{tr } \mathbf{E}^s, n^\ell) \quad (101)$$

where $\text{tr } \mathbf{E}^s$ is the volumetric strain. The volumetric strain can, not only capture the deformation related changes in suction, but also can account for the wetting and drying related hysteresis since volumetric strains produced during wetting will be different from those accumulated during drying. This relationship can be determined by laboratory tests for various soils. In order to account for the influence of initial fabric on h_c , samples of same soil produced by two different compaction techniques, for example static versus dynamic compaction, should be treated as two different soils and two relationships should be established.

Because the properties of interfaces are not of main concern, only 12 unknown variables will be evaluated, namely,

$$n^\ell, n^g, \mathbf{v}^{\ell,s}, \mathbf{v}^{g,s}, \mathbf{E}^s, \rho^\ell, \rho^g, \rho, \mathbf{t}, p^\ell, p^g, \hat{e}_{\ell g}^s(\hat{e}_{\ell g}^s) \quad (102)$$

(87a)–(87c), (90), (93a), (93b) and (101) consist of 7 independent equations. For a closure problem, there are still 5 equations deficient. In most soil mechanics problems, soil grains may be reasonably considered incompressible, therefore

$$D^s \rho^s / Dt = 0 \quad (103a)$$

Incorporating (103a) and (87a) leads to

$$-\frac{D^s n}{Dt} + (1 - n) \nabla \cdot \mathbf{v}^s = 0 \quad (103b)$$

where porosity $n = n^\ell + n^g$. The mass densities of the liquid and the gas are assumed to be a function of the pressure and the absolute temperature of the phase only, i.e.

$$\rho^\alpha = f(p^\alpha, \theta), \quad \alpha = \ell, g \quad (103c)$$

If the change in the density is small (applicable for most geotechnical problems), the truncated Taylor's series expansion of ρ^α provides

$$\frac{1}{\rho^\alpha} \frac{D^s \rho^\alpha}{Dt} = \frac{1}{\rho^\alpha} \frac{\partial \rho^\alpha}{\partial p^\alpha} \frac{D^s p^\alpha}{Dt} + \frac{1}{\rho^\alpha} \frac{\partial \rho^\alpha}{\partial \theta} \frac{D^s \theta}{Dt}, \quad \alpha = \ell, g \quad (104)$$

Defining the bulk modulus k_α as

$$\frac{1}{k_\alpha} = \frac{1}{\rho^\alpha} \frac{\partial \rho^\alpha}{\partial p^\alpha}, \quad \alpha = \ell, g \quad (105a)$$

and thermal expansion coefficient β_α as

$$\beta_\alpha = \frac{1}{\rho^\alpha} \frac{\partial \rho^\alpha}{\partial \theta}, \quad \alpha = \ell, g \quad (105b)$$

(104) may be rewritten as

$$\frac{1}{\rho^\alpha} \frac{D^s \rho^\alpha}{Dt} = \frac{1}{k_\alpha} \frac{D^s p^\alpha}{Dt} + \beta_\alpha \frac{D^s \theta}{Dt}, \quad \alpha = \ell, g \quad (106)$$

For isothermal conditions, the last term of (106) can be simply omitted.

Based on the dissipative entropy inequality (56) and (87d), we can derive the following constitutive relation for the rate of mass transfer, namely,

$$\dot{\epsilon}'_{\text{g}} = -\dot{\epsilon}^{\text{g}}_{\text{g}} = a(G^{\text{g}} - G') \quad (107)$$

where a is a material constant. As an approximation, we propose

$$\dot{\epsilon}'_{\text{g}} = -\dot{\epsilon}^{\text{g}}_{\text{g}} = a(p^{\text{g}} - p^{\ell}) \quad (108)$$

Thus far, all the governing equations for the dynamic behaviour of unsaturated soils have been constructed. These equations include:

- (i) mass balance equations, (87b), (87c), (103a) and (103b),
- (ii) balance equation of linear momentum for the mixture, (90),
- (iii) generalized Darcy's laws (93a) and (93b),
- (iv) balance equation of momentum for interfaces, or suction expression, (101),
- (v) constitutive relation for the rate of mass exchange between liquid and gas, (108),
- (vi) constitutive relation for the deformation of the soil skeleton, (86),
- (vii) the constitutive relationships for the changes in the densities of liquid and gas, (106)

(i)–(vii) include 12 independent equations and 12 independent unknowns. This is a closure problem. Solution of these equations can proceed in a straightforward manner using techniques such as finite element methods, and this work is currently in progress.

SUMMARY AND CONCLUSIONS

Realistic field equations governing the behaviour of unsaturated porous media under dynamic loading conditions are developed using the theory of mixtures with interfaces (TMI). The unsaturated porous media is considered as a mixture consisting of a solid skeleton, a pore liquid, and a pore gas. Unlike the classical theory of mixtures, the TMI considers all the interfaces existing between the bulk phases as separate phases in deriving the balance equations. Macroscopic balance equations in TMI are derived by the local averaging procedure applied to microscopic balance equations. Macroscopic balance equations together with the second law of thermodynamics are used in developing the dynamic governing equations for an unsaturated porous media.

Unlike the previous works that utilized TMI, volume fractions of the bulk phases and area densities of the interfaces are introduced as independent constitutive variables. Volume fractions and area densities, which represent the geometry of the boundary surfaces of the bulk phases, are more fundamental parameters than porosity and degree of saturation used in the previous works. Based on the derivations presented, the following can be concluded.

- (1) A new definition for the total stress is deduced. This definition, in addition to including the partial stresses of bulk phases, includes interfacial surface tensions and area densities.
- (2) Effective stress of the solid skeleton is a function of the total stress, pore gas pressure, pore liquid pressure, interfacial surface tensions and area densities, and volume fractions of the bulk phases, among other things. Since area densities and volume fractions can change during deformations and varies from soil to soil, effective stress becomes dependent on

stress paths and soil types. The above-mentioned phenomenon has been observed previously in experiments and a thermodynamical basis, using the TMI, is presented for the first time in this paper.

- (3) It is found that the Gibbs' thermodynamic potential is the driving potential for the flow of the fluids within the unsaturated materials.
- (4) Instead of a single effective stress, two stress state variables, net stress (total stress minus pore gas pressure) and suction (pore gas pressure minus pore liquid pressure) are more convenient to describe the stress-strain behaviour of the solid skeleton. Phenomenological approach utilizing two stress state variables seem to be the most promising to describe the elastoplastic behaviour of the solid skeleton.
- (5) It is shown that the 'soil-water characteristic curve', which has been borrowed from soil science, intrinsically represents the overall balance of the linear momentum of the interfaces. This finding implies that many formulae prevailing in current practice, which relate soil suction to degree of saturation only, may not necessarily capture the main features of the suction if the soil is allowed to deform.
- (6) The final set of governing equations, developed under certain simplified assumptions, can be readily implemented into techniques like finite element methods to analyse problems such as earthquake loading of compacted soil embankments.

ACKNOWLEDGEMENTS

This research was supported by a grant (Grant No. CMS 9501718) from the National Science Foundation.

NOTATION

| | |
|--------------------------------|---|
| $a^{\alpha\beta}$ | area density of $\alpha\beta$ -interface, L^{-1} |
| A^α | Helmholtz free energy of α -phase, $L^2 T^{-2}$ |
| $A^{\alpha\beta}$ | Helmholtz free energy of $\alpha\beta$ -interface, $L^2 T^{-2}$ |
| b^α | macroscopic external supply of entropy of α -phase, $L^2 T^{-2} \theta^{-1}$ |
| $b^{\alpha\beta}$ | macroscopic external supply of entropy of $\alpha\beta$ -interface, $L^2 T^{-2} \theta^{-1}$ |
| \mathbf{d}^α | symmetric part of velocity gradient of α -phase, T^{-1} |
| $\mathbf{d}^{\alpha\beta}$ | symmetric part of velocity gradient of $\alpha\beta$ -interface, T^{-1} |
| $\hat{e}_{\alpha\beta}^\alpha$ | mass exchange rate from $\alpha\beta$ -interface to α -phase, $M L^{-3} T^{-1}$ |
| $\hat{e}_{s/g}^{\alpha\beta}$ | mass exchange rate from the contact line s/g to $\alpha\beta$ -interface, $M L^{-3} T^{-1}$ |
| E^α | macroscopic internal energy per unit mass of α -phase, $L^2 T^{-2}$ |
| $E^{\alpha\beta}$ | macroscopic internal energy per unit mass of $\alpha\beta$ -interface, $L^2 T^{-2}$ |
| \mathbf{E}^s | Lagrangian stain tensor |
| \mathbf{F}^s | deformation gradient of the solid skeleton |
| \mathbf{g} | gravitational acceleration, $L T^{-2}$ |
| h^α | external supply of energy to α -phase, $L^2 T^{-3}$ |
| $h^{\alpha\beta}$ | external supply of energy to $\alpha\beta$ -interface, $L^2 T^{-3}$ |
| \mathbf{I} | unit tensor |
| n | porosity |
| n^α | volume fraction of α -phase |

| | |
|---|---|
| p^α | thermodynamic pressure of α -phase, $M L^{-1} T^{-2}$ |
| $p_{\alpha\beta}$ | capillary pressure associated with $\alpha\beta$ -interface, $M L^{-1} T^{-2}$ |
| $\bar{p}_{\alpha\beta}$ | dissipative part of capillary pressure, $M L^{-1} T^{-2}$ |
| \mathbf{q}^α | heat flux vector of α -phase, $M T^{-3}$ |
| $\mathbf{q}^{\alpha\beta}$ | heat flux vector of $\alpha\beta$ -interface, $M T^{-3}$ |
| $\bar{Q}_{\alpha\beta}^\alpha$ | body supply of heat to α -phase from $\alpha\beta$ -interface, $M L^{-1} T^{-3}$ |
| $\bar{Q}_{s/g}^{\alpha\beta}$ | body supply of heat to $\alpha\beta$ -interface from the contact line s/g , $M L^{-1} T^{-3}$ |
| $\mathbf{S}^{\alpha\beta}$ | macroscopic stress tensor of $\alpha\beta$ -interface, $M T^{-2}$ |
| $\hat{\mathbf{S}}_{s/g}^{\alpha\beta}$ | body supply of momentum to the $\alpha\beta$ -interface from the contact line s/g , $M L^{-2} T^{-2}$ |
| S_c | suction, i.e. capillary pressure associated with the contractile skin, $M L^{-1} T^{-2}$ |
| \mathbf{t} | total stress tensor of the mixture, $M L^{-1} T^{-2}$ |
| $\mathbf{t} _\alpha$ | microscopic stress tensor of α -phase, $M L^{-1} T^{-2}$ |
| \mathbf{t}' | effective stress tensor of the solid skeleton, $M L^{-1} T^{-2}$ |
| \mathbf{t}^α | macroscopic stress tensor of α -phase, $M L^{-1} T^{-2}$ |
| $\hat{\mathbf{T}}_{\alpha\beta}^\alpha$ | body supply of momentum to α -phase from $\alpha\beta$ -interface, $M L^{-2} T^{-2}$ |
| \mathbf{v}^α | macroscopic velocity of α -phase phase, $L T^{-1}$ |
| $\tilde{\mathbf{v}}^\alpha$ | diffusion velocity of α -phase, $L T^{-1}$ |
| $\mathbf{v} _\alpha$ | microscopic velocity of $\alpha\beta$ -interface, $L T^{-1}$ |
| $\mathbf{w}^{\alpha\beta}$ | macroscopic velocity of $\alpha\beta$ -interface, $L T^{-1}$ |
| \mathbf{x} | position vector of the solid phase is deformed configuration, L |
| \mathbf{X}^α | position vector of a material point of α -phase in undeformed configuration, L |

Greek Letters

| | |
|----------------------------|---|
| $\Gamma^{\alpha\beta}$ | surface excess mass density, $M L^{-2}$ |
| $\gamma^{\alpha\beta}$ | macroscopic interfacial tension, $M T^{-2}$ |
| γ_v | equivalent averaged radius of air bubbles, L |
| δ | coefficient of curvature variation of the contractile skin |
| η^α | macroscopic internal entropy per unit mass of α -phase, $L^2 T^{-2} \theta^{-1}$ |
| $\eta^{\alpha\beta}$ | macroscopic internal entropy per unit mass of $\alpha\beta$ -interface, $L^2 T^{-2} \theta^{-1}$ |
| θ | temperature, θ |
| Λ | entropy |
| ρ | overall (volumetric) mass density of the mixture, $M L^{-3}$ |
| ρ^α | intrinsic (volumetric) mass density of α -phase, $M L^{-3}$ |
| $\bar{\tau}^\alpha$ | non-equilibrium part of the linear momentum exchange of α -phase, $M L^{-2} T^{-2}$ |
| $\bar{\tau}^{\alpha\beta}$ | non-equilibrium part of the linear momentum exchange of $\alpha\beta$ -interface, $M L^{-2} T^{-2}$ |
| ϕ^α | macroscopic entropy flux vector of α -phase, $M T^{-3} \theta^{-1}$ |
| $\phi^{\alpha\beta}$ | macroscopic entropy flux vector of $\alpha\beta$ -interface, $M T^{-3} \theta^{-1}$ |

Special notation

| | |
|---------------|---|
| ∇ | gradient of a quantity with respect to the deformed configuration \mathbf{x} , L^{-1} |
| D^α/Dt | material derivative following the motion of α -phase, T^{-1} |
| GRAD | gradient of a quantity with respect to the reference frame \mathbf{X} , L^{-1} |
| g | gas |
| ℓ | liquid |

| | |
|-------------------|--|
| s | solid |
| α | α bulk phase |
| $\alpha\beta$ | $\alpha\beta$ -interface |
| $ _\alpha$ | microscopic value of a quantity in α -phase |
| $()_{\text{eq}}$ | terms inside parentheses evaluated at equilibrium |

APPENDIX I

The restrictions of thermodynamics

A general entropy inequality has been derived by Hassanizadeh and Gray,³⁵ namely

$$\begin{aligned}
 \Lambda = & -\sum_{\alpha} \frac{n^{\alpha} \rho^{\alpha}}{\theta^{\alpha}} \left(\frac{D^{\alpha} A^{\alpha}}{Dt} + \eta^{\alpha} \frac{D^{\alpha} \theta^{\alpha}}{Dt} \right) + \sum_{\alpha} \frac{n^{\alpha}}{(\theta^{\alpha})^2} \mathbf{q}^{\alpha} \cdot \nabla \theta^{\alpha} + \sum_{\alpha} \frac{n^{\alpha}}{\theta^{\alpha}} \mathbf{t}^{\alpha} : \mathbf{d}^{\alpha} \\
 & - \sum_{\alpha\beta} \frac{a^{\alpha\beta} \Gamma^{\alpha\beta}}{\theta^{\alpha\beta}} \left(\frac{D^{\alpha\beta} A^{\alpha\beta}}{Dt} + \eta^{\alpha\beta} \frac{D^{\alpha\beta} \theta^{\alpha\beta}}{Dt} \right) + \sum_{\alpha\beta} \frac{a^{\alpha\beta}}{(\theta^{\alpha\beta})^2} \mathbf{q}^{\alpha\beta} \cdot \nabla \theta^{\alpha\beta} + \sum_{\alpha\beta} \frac{a^{\alpha\beta}}{\theta^{\alpha\beta}} \mathbf{S}^{\alpha\beta} : \mathbf{d}^{\alpha\beta} \\
 & - \frac{1}{\theta^s} \sum_{\alpha\beta} \left[\frac{\hat{Q}_{s/\alpha\beta}^{\alpha\beta} \theta^{\alpha\beta, s}}{\theta^{\alpha\beta}} + \hat{\mathbf{S}}_{s/\alpha\beta}^{\alpha\beta} \cdot \mathbf{w}^{\alpha\beta, s} + \hat{e}_{s/\alpha\beta}^{\alpha\beta} \left(A^{\alpha\beta, s} + \frac{1}{2} (w^{\alpha\beta, s})^2 + \eta^{\alpha\beta} \theta^{\alpha\beta, s} \right) \right] \\
 & + \sum_{\alpha} \sum_{\beta \neq \alpha} \frac{1}{\theta^{\alpha\beta}} \left[\frac{\hat{Q}_{\alpha\beta}^{\alpha} \theta^{\alpha\beta, \alpha}}{\theta^{\alpha}} + \hat{\mathbf{T}}_{\alpha\beta}^{\alpha} \cdot \mathbf{w}^{\alpha\beta, \alpha} + \hat{e}_{\alpha\beta}^{\alpha} \left(A^{\alpha\beta, \alpha} - \frac{1}{2} (w^{\alpha\beta, \alpha})^2 + \eta^{\alpha} \theta^{\alpha\beta, \alpha} \right) \right] \geq 0
 \end{aligned} \tag{109}$$

Applying chain rule to constitutive assumptions (34a)–(34d), we have

$$\begin{aligned}
 \frac{D^{\ell} A^{\ell}}{Dt} = & \frac{\partial A^{\ell}}{\partial \rho^{\ell}} \cdot \frac{D^{\ell} \rho^{\ell}}{Dt} + \frac{\partial A^{\ell}}{\partial \theta^{\ell}} \cdot \frac{D^{\ell} \theta^{\ell}}{Dt} \\
 & + \frac{\partial A^{\ell}}{\partial n^{\ell}} \cdot \frac{D^{\ell} n^{\ell}}{Dt} + \frac{\partial A^{\ell}}{\partial a^{\ell g}} \cdot \frac{D^{\ell} a^{\ell g}}{Dt} + \frac{\partial A^{\ell}}{\partial a^{\ell s}} \cdot \frac{D^{\ell} a^{\ell s}}{Dt}
 \end{aligned} \tag{110}$$

$$\begin{aligned}
 \frac{D^g A^g}{Dt} = & \frac{\partial A^g}{\partial \rho^g} \cdot \frac{D^g \rho^g}{Dt} + \frac{\partial A^g}{\partial \theta^g} \cdot \frac{D^g \theta^g}{Dt} \\
 & + \frac{\partial A^g}{\partial n^g} \cdot \frac{D^g n^g}{Dt} + \frac{\partial A^g}{\partial a^{\ell g}} \cdot \frac{D^g a^{\ell g}}{Dt} + \frac{\partial A^g}{\partial a^{sg}} \cdot \frac{D^g a^{sg}}{Dt}
 \end{aligned} \tag{111}$$

$$\begin{aligned}
 \frac{D^s A^s}{Dt} = & \frac{\partial A^s}{\partial \rho^s} \cdot \frac{D^s \rho^s}{Dt} + \frac{\partial A^s}{\partial \theta^s} \cdot \frac{D^s \theta^s}{Dt} \\
 & + \frac{\partial A^s}{\partial \mathbf{E}^s} \cdot \frac{D^s \mathbf{E}^s}{Dt} + \frac{\partial A^s}{\partial a^{\ell s}} \cdot \frac{D^s a^{\ell s}}{Dt} + \frac{\partial A^s}{\partial a^{sg}} \cdot \frac{D^s a^{sg}}{Dt}
 \end{aligned} \tag{112}$$

$$\begin{aligned}
 \frac{D^{\alpha\beta} A^{\alpha\beta}}{Dt} = & \frac{\partial A^{\alpha\beta}}{\partial \Gamma^{\alpha\beta}} \cdot \frac{D^{\alpha\beta} \Gamma^{\alpha\beta}}{Dt} + \frac{\partial A^{\alpha\beta}}{\partial \theta^{\alpha\beta}} \cdot \frac{D^{\alpha\beta} \theta^{\alpha\beta}}{Dt} \\
 & + \frac{\partial A^{\alpha\beta}}{\partial n^{\ell}} \cdot \frac{D^{\alpha\beta} n^{\ell}}{Dt} + \frac{\partial A^{\alpha\beta}}{\partial n^g} \cdot \frac{D^{\alpha\beta} n^g}{Dt} + \frac{\partial A^{\alpha\beta}}{\partial a^{\alpha\beta}} \cdot \frac{D^{\alpha\beta} a^{\alpha\beta}}{Dt}
 \end{aligned} \tag{113}$$

Substituting (110)–(113) into (109) and incorporating mass balance equations (16a) and (16b) yield the following restrictions of thermodynamics for the proposed constitutions:

$$\begin{aligned}
\Lambda = & -\sum_{\alpha} \frac{n^{\alpha} \rho^{\alpha}}{\theta^{\alpha}} \frac{D^s \theta^{\alpha}}{Dt} \left(\frac{\partial A^{\alpha}}{\partial \theta^{\alpha}} + \eta^{\alpha} \right) \\
& - \sum_{\alpha\beta} \frac{a^{\alpha\beta} \Gamma^{\alpha\beta}}{\theta^{\alpha\beta}} \frac{D^s \theta^{\alpha\beta}}{Dt} \left(\frac{\partial A^{\alpha\beta}}{\partial \theta^{\alpha\beta}} + \eta^{\alpha\beta} \right) \\
& + \sum_{\alpha \neq s} \frac{n^{\alpha}}{\theta^{\alpha}} \mathbf{d}^{\alpha} : (p^{\alpha} \mathbf{I} + \mathbf{t}^{\alpha}) \\
& + \frac{n^s}{\theta^s} \mathbf{d}^s : \left[p^s \mathbf{I} + \mathbf{t}^s - \rho \mathbf{F}^{s^T} \frac{\partial A^s}{\partial \mathbf{E}^s} \mathbf{F}^s \right] \\
& + \sum_{\alpha \neq s} \dot{n}^{\alpha} \left(\frac{p^{\alpha}}{\theta^{\alpha}} - \frac{p^s}{\theta^s} - \frac{n^{\alpha} \rho^{\alpha}}{\theta^{\alpha}} \frac{\partial A^{\alpha}}{\partial n^{\alpha}} - \sum_{\beta\gamma} \frac{a^{\beta\gamma} \Gamma^{\beta\gamma}}{\theta^{\beta\gamma}} \frac{\partial A^{\beta\gamma}}{\partial n^{\alpha}} \right) \\
& + \sum_{\alpha \neq s} \frac{\mathbf{v}^{\alpha,s}}{\theta^{\alpha}} \cdot \left[\left(p^{\alpha} - n^{\alpha} \rho^{\alpha} \frac{\partial A^{\alpha}}{\partial n^{\alpha}} \right) \nabla n^{\alpha} - \sum_{\beta \neq \alpha} n^{\alpha} \rho^{\alpha} \frac{\partial A^{\alpha}}{\partial a^{\alpha\beta}} \nabla a^{\alpha\beta} \right. \\
& \quad \left. - n^{\alpha} \rho^{\alpha} \frac{\partial A^{\alpha}}{\partial \theta^{\alpha}} \nabla \theta^{\alpha} - \sum_{\beta \neq \alpha} \hat{\mathbf{T}}_{\alpha\beta}^{\alpha} + \sum_{\beta \neq \alpha} \hat{\mathbf{T}}_{\alpha\beta}^{\alpha} \frac{\theta^{\alpha\beta,\alpha}}{\theta^{\alpha\beta}} \right] + \sum_{\alpha \neq s} \frac{a^{\alpha\beta}}{\theta^{\alpha\beta}} \mathbf{d}^{\alpha\beta} : (\mathbf{S}^{\alpha\beta} - \gamma^{\alpha\beta} \mathbf{I}) \\
& + \sum_{\alpha\beta} \frac{\mathbf{w}^{\alpha\beta,s}}{\theta^{\alpha\beta}} \cdot \left[\left(-\gamma^{\alpha\beta} - a^{\alpha\beta} \Gamma^{\alpha\beta} \frac{\partial A^{\alpha\beta}}{\partial a^{\alpha\beta}} \right) \nabla a^{\alpha\beta} - a^{\alpha\beta} \Gamma^{\alpha\beta} \frac{\partial A^{\alpha\beta}}{\partial n^{\alpha}} \nabla n^{\alpha} \right. \\
& \quad \left. - a^{\alpha\beta} \Gamma^{\alpha\beta} \frac{\partial A^{\alpha\beta}}{\partial n^s} \nabla n^s - a^{\alpha\beta} \Gamma^{\alpha\beta} \frac{\partial A^{\alpha\beta}}{\partial \theta^{\alpha\beta}} \nabla \theta^{\alpha\beta} - \hat{\mathbf{S}}_{s/g}^{\alpha\beta} + \hat{\mathbf{T}}_{\alpha\beta}^{\alpha} + \hat{\mathbf{T}}_{\alpha\beta}^{\beta} - \hat{\mathbf{S}}_{s/g}^{\alpha\beta} \frac{\theta^{\alpha\beta,s}}{\theta^s} \right] \\
& + \sum_{\alpha} \frac{n^{\alpha} \mathbf{q}^{\alpha}}{(\theta^{\alpha})^2} \cdot \nabla \theta^{\alpha} + \sum_{\alpha\beta} \frac{a^{\alpha\beta} \mathbf{q}^{\alpha\beta}}{(\theta^{\alpha\beta})^2} \cdot \nabla \theta^{\alpha\beta} \\
& - \sum_{\alpha\beta} \dot{a}^{\alpha\beta} \left(\frac{n^{\alpha} \rho^{\alpha}}{\theta^{\alpha}} \frac{\partial A^{\alpha}}{\partial a^{\alpha\beta}} + \frac{n^{\beta} \rho^{\beta}}{\theta^{\beta}} \frac{\partial A^{\beta}}{\partial a^{\alpha\beta}} + \frac{\gamma^{\alpha\beta}}{\theta^{\alpha\beta}} + \frac{a^{\alpha\beta} \Gamma^{\alpha\beta}}{\theta^{\alpha\beta}} \frac{\partial A^{\alpha\beta}}{\partial a^{\alpha\beta}} \right) \\
& - \sum_{\alpha\beta} \frac{\hat{Q}_{\alpha\beta\gamma}^{\alpha\beta} \theta^{\alpha\beta,s}}{\theta^s \theta^{\alpha\beta}} + \sum_{\alpha} \sum_{\beta \neq \alpha} \frac{\hat{Q}_{\alpha\beta}^{\alpha} \theta^{\alpha\beta,\alpha}}{\theta^{\alpha} \theta_{\alpha\beta}} \\
& + \sum_{\alpha} \sum_{\beta \neq \alpha} \frac{\hat{e}_{\alpha\beta}^{\alpha}}{\theta^{\alpha} \theta^{\alpha\beta}} \left\{ \left(\eta^{\alpha} \theta^{\alpha} - \frac{p^{\alpha}}{\rho^{\alpha}} \right) \theta^{\alpha\beta,\alpha} + \left[G^{\alpha\beta,\alpha} - \frac{1}{2} (w^{\alpha\beta,\alpha})^2 \right] \theta^{\alpha} \right\} \\
& + \sum_{\alpha\beta} \frac{\hat{e}_{s/g}^{\alpha\beta}}{\theta^s} \left[G^{\alpha\beta,s} + \frac{1}{2} (w^{\alpha\beta,s})^2 + \frac{\theta^{\alpha\beta,s}}{\theta^{\alpha\beta}} \left(\eta^{\alpha\beta} \theta^{\alpha\beta} - \frac{\gamma^{\alpha\beta}}{\Gamma^{\alpha\beta}} \right) \right] \\
& \geq 0
\end{aligned} \tag{114}$$

where G^α and $G^{\alpha\beta}$ are the Gibbs' thermodynamic potentials defined by

$$G^\alpha = A^\alpha + p^\alpha/\rho^\alpha \quad (115)$$

$$G^{\alpha\beta} = A^{\alpha\beta} - \gamma^{\alpha\beta}/\Gamma^{\alpha\beta} \quad (116)$$

In (114) entropy Λ is a linear function of variables $D^s\theta^\alpha/Dt$, $D^s\theta^{\alpha\beta}/Dt$, \mathbf{d}^α , and $\mathbf{d}^{\alpha\beta}$, on the other hand, no constitutive relations were assumed to be the function of these variables. Following the arguments of Coleman and Noll,⁵⁵ at least there exists one thermodynamic process for which variables $D^s\theta^\alpha/Dt$, $D^s\theta^{\alpha\beta}/Dt$, \mathbf{d}^α , and $\mathbf{d}^{\alpha\beta}$ can be arbitrary values. Therefore, if the inequality (114) is valid for all possible thermodynamic events, the coefficients of these variables must vanish, therefore we have

$$\eta^\alpha + \partial A^\alpha/\partial\theta^\alpha = 0 \quad (117a)$$

$$\eta^{\alpha\beta} + \partial A^{\alpha\beta}/\partial\theta^{\alpha\beta} = 0 \quad (117b)$$

$$\mathbf{t}^\alpha + p^\alpha \mathbf{I} = 0 \quad (118)$$

$$\mathbf{S}^{\alpha\beta} - \gamma^{\alpha\beta} \mathbf{I} = 0 \quad (119)$$

$$\mathbf{t}^s + p^s \mathbf{I} - \mathbf{t}^e = 0 \quad (120)$$

where

$$p^\alpha = (\rho^\alpha)^2 \partial A^\alpha/\partial\rho^\alpha \quad (121a)$$

$$\gamma^{\alpha\beta} = -(\Gamma^{\alpha\beta})^2 \partial A^{\alpha\beta}/\partial\Gamma^{\alpha\beta} \quad (121b)$$

$$\mathbf{t}^e = \rho^s \mathbf{F}^{s\tau} \frac{\partial A^s}{\partial \mathbf{E}^s} \mathbf{F}^s \quad (121c)$$

In defining (121b), $\dot{a}^{\alpha\beta}$ has been used as a dependent variable. If $\dot{\Gamma}^{\alpha\beta}$ is used instead of $\dot{a}^{\alpha\beta}$, we have another definition, i.e.

$$\gamma^{\alpha\beta} = -a^{\alpha\beta} \Gamma^{\alpha\beta} \frac{\partial A^{\alpha\beta}}{\partial a^{\alpha\beta}} \quad (122)$$

REFERENCES

1. M. A. Biot, 'General theory of three-dimensional consolidation', *J. Appl. Phys.*, **12**, 155–164 (1941).
2. M. A. Biot, 'Mechanics of deformation and acoustic propagation in porous media', *J. Appl. Phys.*, **33**, 1482–1498 (1956).
3. O. C. Zienkiewicz and T. Shiomi, 'Dynamic behaviour of saturated porous media: the generalized Biot formulation and its numerical solution', *Int. J. Numer. Anal. Meth. Geomech.*, **8**, 71–96 (1984).
4. J. H. Prevost, 'Mechanics of continuous porous media', *Int. J. Engng. Sci.*, **81**, 787–800 (1980).
5. J. H. Prevost, 'Nonlinear transient phenomena in saturated porous media', *Comput. Meth. Appl. Mech. Engng.*, **30**, 3–18 (1982).
6. K. K. Muraleetharan, K. D. Mish and K. Arulanandan, 'A fully coupled nonlinear dynamic analysis procedure and its verification using centrifuge test results', *Int. J. Numer. Anal. Meth. Geomech.*, **18**, 305–325 (1994).
7. D. G. Fredlund, 'Appropriate concepts and technology for unsaturated soils', *Can. Geotech. J.*, **16**, 121–139 (1979).
8. D. G. Fredlund and N. R. Morgenstern, 'Stress state variables for unsaturated soils', *J. Geotech. Engng. ASCE*, **103**(GT5), 447–466 (1977).
9. D. G. Fredlund, N. R. Morgenstern and R. A. Wiger, 'The shear strength of unsaturated soils', *Can. Geotech. J.*, **45**(3), 313–321 (1978).

10. C. Truesdell and R. Toupin, in S. Flugge (ed.), *The Classical Field Theories, Handbook der Physik*, Vol. III/1, Springer, Berlin, 1960.
11. R. M. Bowen, 'Compressible porous media models by use of the theory of mixtures', *Int. J. Engng. Sci.*, **20**(6), 697–735 (1982).
12. H. Demiryay, 'A continuum theory of chemically reacting mixtures of fluids and solids', *Int. J. Engng. Sci.*, **19**, 253–268 (1981).
13. L. Thigpen and J. G. Berryman, 'Mechanics of porous elastic materials containing multiphase fluid', *Int. J. Engng. Sci.*, **23**(23), 1203–1214 (1986).
14. A. C. Eringen, 'A continuum theory of swelling porous elastic soils', *Int. J. Engng. Sci.*, **32**(8), 1337–1349 (1994).
15. X. Li and X. Li, 'On the thermoelasticity of multicomponent fluid-saturated reacting porous media', *Int. J. Engng. Sci.*, **30**(7), 891–912 (1992).
16. X. Li, 'Elastic coefficients for liquid- and gas-saturated porous media', *Int. J. Engng. Sci.*, **32**(2), 195–208 (1994).
17. A. Verruijt, 'The theory of consolidation', *Proc. NATO Adv. Study Inst. On Mechanics of Fluids in Porous Media*, Martinus Nijhoff, the Hague, The Netherlands, 1982.
18. I. Vardoulakis and D. G. Beskos, 'Dynamic behavior of nearly saturated porous media', *Mech. Mater.*, **5**, 87–108 (1986).
19. C. S. Chang and J. M. Duncan, 'Consolidation analysis for partly saturated clay by using an elastic-plastic effective-strain model', *Int. J. Numer. Anal. Mech. Geomech.*, **7**, 39–55 (1983).
20. Z.-H. Chen, D.-Y. Xie and Z.-D. Liu, 'Consolidation theory of unsaturated soil based on the theory of mixture (I)', *Appl. Math. Mech.* (English Edition), **14**(2), 137–150 (1993).
21. I. Vardoulakis and D. G. Beskos, 'Dynamic consolidation of nearly saturated granular media', in P. K. Banerjee and R. Butterfield (eds), *Dynamics of Behavior of Foundations and Buried Structures: Developments in Soil Mechanics and Foundation Engineering-3*, Elsevier Applied Science, London, 1987.
22. B. A. Schrefler, 'Mechanics of partially saturated porous media', in C. S. Desai and G. Gioda (eds), *Numerical Methods and Constitutive Modeling in Geomechanics*, CISM Lecture Notes, Springer-Verlag, Wien, 1990, pp. 169–209.
23. O. C. Zienkiewicz, Y. M. Xie, B. A. Schrefler, A. Ledesma, and N. Bicanic, 'Static and dynamic behavior of soils: a rational approach to quantitative solutions. II semi-saturated problems', *Proc. Roy. Soc. London A*, **429**, 311–321 (1990).
24. O. C. Zienkiewicz and Y. M. Xie, 'Analysis of the lower San Fernando dam failure under earthquake', *Dam Engng.*, **II**(4), 307–322 (1991).
25. X. Li and O. C. Zienkiewicz, 'Multiphase flow in deforming porous media and finite element solutions', *Comput. Struct.*, **45**(2), 211–227 (1992).
26. B. A. Schrefler and X. Zhan, 'A fully coupled model for water flow and airflow in deformable porous media', *Water Resour. Res.*, **29**, 155–167 (1993).
27. E. A. Meroi and B. A. Schrefler and O. C. Zienkiewicz, 'Large strain static and dynamic semi-saturated soil behavior', *Int. J. Numer. Anal. Mech. Geomech.*, **19**, 81–106 (1995).
28. D. Gawin, and B. A. Schrefler, 'Thermo-hydro-mechanical analysis of partially saturated porous materials', *Engng. Comput.*, **13**(7), 113–143 (1996).
29. A. W. Bishop, 'The principle of effective stress', Lectures delivered in Oslo, Norway, 1955, published in *Tek. Ukebl.*, **106**(39), 859–863 (1959).
30. D. G. Fredlund and H. Rahardjo, *Soil Mechanics for Unsaturated Soils*, Wiley, New York, 1993.
31. A. Gens, 'Constitutive modeling: application to compacted soils', in Alonso and Delago (eds), *Proc. 1st Int. Conf. on Unsaturated Soils, Paris, France, 1995*, Vol. III, A. A. Balkema, Rotterdam, Netherlands, 1996.
32. K. K. Muraleetharan and P. R. Nedunuri, 'A bounding surface elastoplastic constitutive model for monotonic and cyclic behavior of unsaturated soils', *Proc. 12th Engineering Mechanics Conf.*, ASCE, La Jolla, CA, 1998, pp. 1331–1334.
33. S. M. Hassanizadeh and W. G. Gray, 'General conservation equations for multiphase systems: a & b', *Adv. Water Resour.*, **2**, 131–203 (1979).
34. W. G. Gray and S. M. Hassanizadeh, 'Averaging theorems and averaged equations for transport of interface properties in multiphase systems', *Int. J. Multiphase Flow*, **15**(1), 81–95 (1989).
35. S. M. Hassanizadeh and W. G. Gray, 'Mechanics and thermodynamics of multiphase flow in porous media including interphase boundaries', *Adv. Water Resour.*, **13**(4), 169–186 (1990).
36. W. G. Gray and S. M. Hassanizadeh, 'Unsaturated flow theory including interfacial phenomena', *Water Resour. Res.*, **27**(8), 1855–1853 (1991).
37. J. H. Prevost, 'Nonlinear dynamic response analysis of soil and soil structure interacting systems', *Soil Dynamics and Geotechnical Earthquake Engineering*, A. A. Balkema, Rotterdam, Netherlands, 1993.
38. C. A. Miller and P. Neogi, *Interfacial Phenomena: Equilibrium and Dynamic Effects*, Marcel Dekker, New York, 1985.
39. D. A. Drew and R. T. Lahey, 'Application of general constitutive principles to the derivation of multidimensional two-phase flow equations', *Int. J. Engng. Sci.*, **5**, 243–264 (1979).
40. L. W. Morland, 'A simple constitutive theory for a fluid-saturated porous solid', *J. Geophys. Res.*, **77**(5), 890–900 (1972).

41. A. Bedford and D. S. Drumheller, 'Recent advances: theories of immiscible and structured mixtures', *Int. J. Engng. Sci.*, **21**(8), 863–960 (1980).
42. D. Drumheller and A. Bedford, 'A thermomechanical theory for reacting immiscible mixtures', *Arch. Rational Mech. Anal.*, **73**, 257–284 (1980).
43. S. R. Degroot and P. Mazur, *Nonequilibrium Thermodynamics*, Wiley (Interscience), New York, 1962.
44. S. Achanta and J. H. Cushman, 'Nonequilibrium swelling- and capillary-pressure relations for colloidal systems', *J. Colloid. Interface Sci.*, **168**, 266–268 (1994).
45. M. A. Goodman and S. C. Cowin, 'A continuum theory for granular materials', *Arch. Rational Mech. Anal.*, **44**, 249–266 (1972).
46. J. W. Nunziato and S. C. Cowin, 'A nonlinear theory of elastic materials with voids', *Arch. Rational Mech. Anal.*, **72**, 177–201 (1979).
47. S. Achanta, J. H. Cushman and M. R. Okos, 'On multicomponent, multiphase thermodynamics', *Int. J. Engng. Sci.*, **32**(11), 1717–1738 (1994).
48. J. C. Muccino, W. G. Gray and L. A. Ferrand, 'Toward an improved understanding of multiphase flow in porous media', *Adv. Geophys.*, **36**(3), 401–422 (1998).
49. K. Terzaghi, 'The shear resistance of saturated soils', in *Proc. 1st Int. Conf. Soil Mech. Found. Engng.*, Vol. 1, Cambridge, MA, 1936, pp. 54–56.
50. O. C. Zienkiewicz, 'Basic formulation of static and dynamic behavior of soil and other porous media', in J. B. Martins (eds), *Numerical Methods in Geomechanics*, Reidal, Company, Dordrecht, 1982.
51. E. C. Matyas and H. S. Radhakrishna, 'Volume change characteristics of partially saturated soils', *Geotechnique*, **18**, 432–448 (1968).
52. S. Pietruszczak and G. N. Pande, 'Constitutive Relation for partially saturated soils containing gas inclusion', *J. Geotech. Engng. ASCE*, **122**(1), (1996).
53. G. H. Crapiste, E. Rotstein and S. Whitaker, 'A general closure schedule for the method of volume averaging', *Chem. Engng. Sci.*, **41**(2), 765–780 (1994).
54. W. G. Gray, A. Leijnse, R. L. Kolar and C. A. Blain, *Mathematical Tool for Changing Spatial Scales in the Analysis of Physical Systems*, CRC Press, Boca Raton, FL, 1993.
55. B. D. Coleman and W. Noll, 'The thermodynamics of elastic materials with heat conduction and viscosity', *Arch. Rational Mech. Anal.*, **13**, 167–184 (1963).